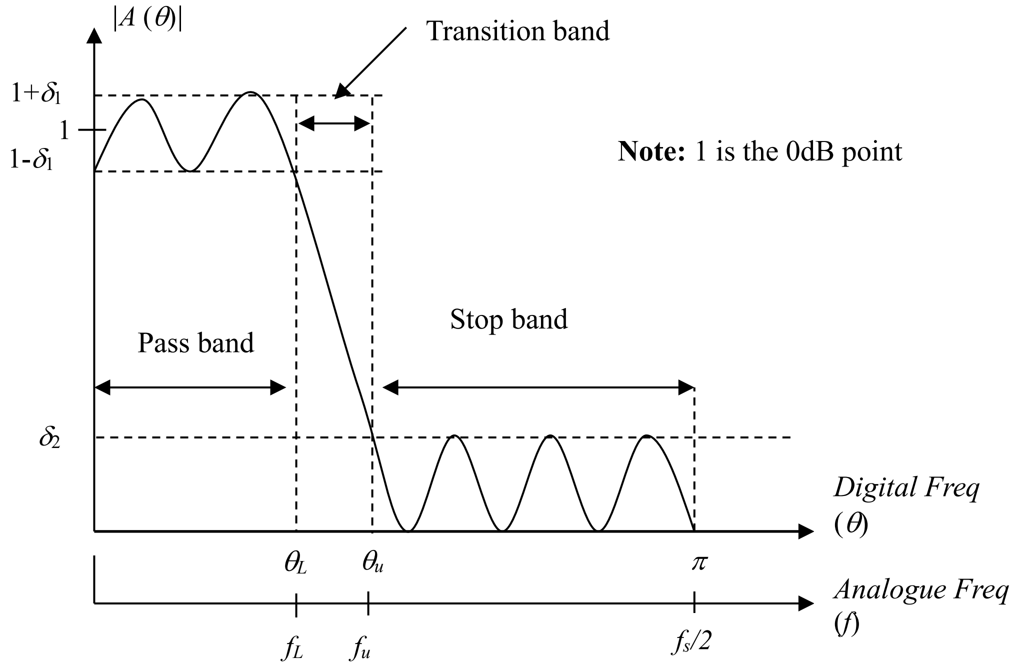
Chapter 8

Chapter 8: Digital Filter Design

8.1 Introduction

The design of a digital filter is the task of determining a transfer function which is a rational function of z^{-1} (e.g. $\frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}}$) in the case of a recursive filter (IIR) or a polynomial in z^{-1}
($a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}$) in the case of a non-recursive filter.

Performance specifications: A typical amplitude characteristic of a low-pass filter is shown below.



The goal of the design is to determine a transfer function H(z) so that its amplitude characteristic |H(θ)| satisfies the conditions.

$$1 - \delta_1 \le |H(\theta)| \le 1 + \delta_1 \quad \text{for } 0 \le \theta \leqslant \theta_L$$
$$|H(\theta)| \le \delta_2 \quad \text{for } \theta_u \le \theta \le \pi$$

8.2 Choosing between FIR and IIR filters

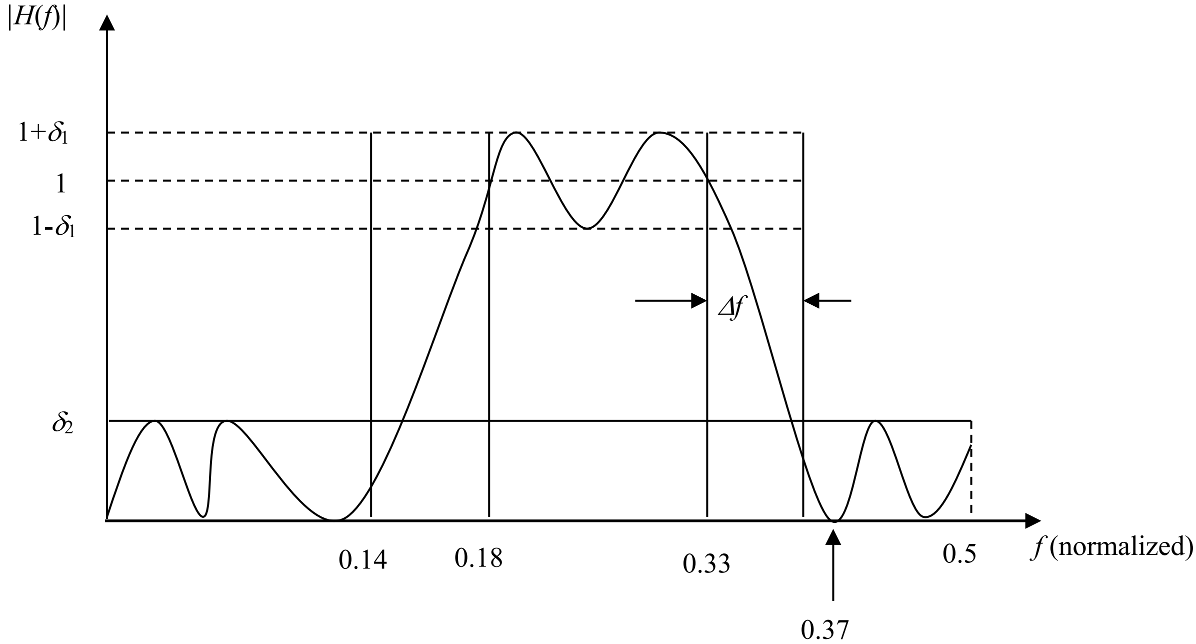
FIR Filters	IIR Filters
Difference equation $y[n] = b_0 x[n] + b_1 x[n-1] + ... + b_M x[n-M]$	Difference equation $y[n] = b_0 x[n] + b_1 x[n-1] + ... + b_M x[n-M]$ $- a_1 y[n-1] - ... - a_N y[n-N]$
Transfer function $H(z) = b_0 + b_1 z^{-1} + ... + b_M z^{-M}$	Transfer function $H(z) = \frac{b_0 + b_1 z^{-1} + ... + b_M z^{-M}}{1 + a_1 z^{-1} + ... + a_N z^{-N}}$
System function contain only zeros	Contain poles and zeros (normally)
Non-recursive or recursive structures are both possible; the best known is the non-recursive (transversal) structure.	Only recursive structure is possible; the most widely used form is the cascade connection of first-order and second order sections.
FIR Filters can have an exactly linear phase response. The implication of this is that no phase distortion is introduced into the signal by the filter.	The phase responses of IIR filters are nonlinear, especially at the band edges.
The effects of using a limited number of bits to implement filters such as round off noise and quantization errors are much less severe in FIR than in IIR.	Because of quantization of the filter coefficients, a pole can in principle move from a position inside the unit circle to a position outside the unit circle and hence cause instability.
FIR requires more coefficients for sharp cut-off filters than IIR. Thus for a given amplitude response specification, more processing time and storage will be required for FIR implementation.	IIR requires fewer coefficients for sharp cut off filters than FIR.
Complexity is proportional to the length of the impulse response.	No direct relation between the complexity and the length of the impulse response (which is infinite by definition) Filters with high selectivity can be realized with relatively low complexity.
FIR filters have no analogue counterpart. FIR design procedures are normally iterative procedures. Design equations do not exist.	Analogue filters can be readily transformed into equivalent IIR digital filters meeting similar specifications. IIR filters can be designed using design formulae.

Example: An FIR filter is to be designed to meet the following frequency response specifications.

- Pass band 0.18-0.33 (normalized)
- Transition band 0.04 (normalized)
- Stop-band deviation 0.001
- passband deviation 0.05

(i) Sketch the tolerance scheme for the filter.
(ii) Express the filter band edge frequencies in the standard unit of kHz, assuming a sampling frequency of 10 kHz and the stop band and pass band deviation in dBs.

The tolerance diagram for the filter is shown below:



f_s=10kHz , therefore

Pass band: 1.8-3.3 kHz
Stop band: 0-1.4 kHz and 3.7-5kHz
Stop band attenuation: $-20\log_{10}(0.001) = -60$dB
Pass band ripple: $20\log_{10}(1+0.05) = 0.42$ dB

Example: The following transfer functions represent two different filters meeting identical amplitude–frequency response specifications.

$$H_1(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}} \longleftarrow \quad \text{Filter 1}$$

where $a_0 = 0.4981819$; $a_1 = 0.9274777$;
$a_2 = 0.4981819$; $b_1 = -0.6744878$; $b_2 = -0.3633482$;

$$H(z) = \sum_{k=0}^{11} h[k] z^{-k} \longleftarrow \quad \text{Filter 2}$$

where

$$h[0] = 0.54603280 \times 10^{2} = h[11]$$

$$h[1] = -0.45068750 \times 10^{-1} = h)[10]$$

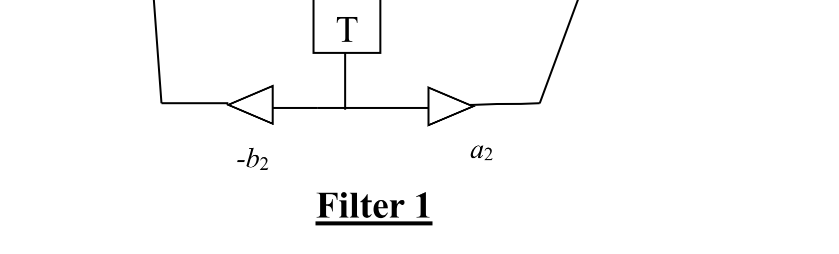
$$h[2] = 0.69169420 \times 10^{-1} = h[9]$$

$$h[3] = -0.55384370 \times 10^{-1} = h[8]$$

$$h[4] = -0.63428410 \times 10^{-1} = h[7]$$

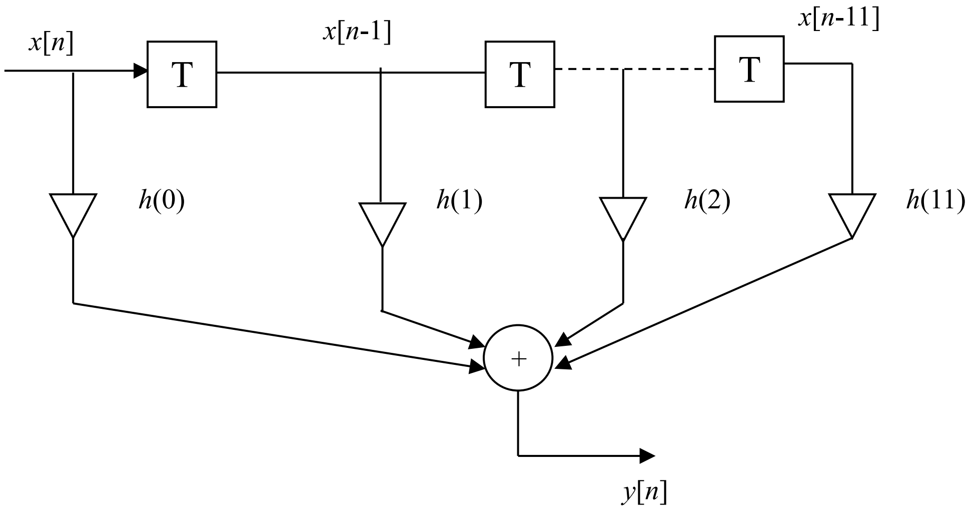
$$h[5] = 0.57892400 \times 10^{0} = h[6]$$

Determine and comment on the computational and storage requirements.



Filter 1

$$w(n) = x(n) - b_1 w(n-1) - b_2 w(n-2)$$
$$y(n) = a_o w(n) + a_1 w(n-1) + a_2 w(n-2)$$



Filter 2

Computation and storage requirement of FIR and IIR

	Filter 1 (IIR)	Filter 2 (FIR)
Number of Multiplications	5	12
Number of Additions	4	11
Storage locations for coefficients and data	8	24

It is evident that the IIR filter is more economical in both computations and storage requirements than the FIR filter.

Exercise:

Determine the computional and storage requirements of the following filter

$$H(z) = \frac{0.8 - 0.5z^{-2} + 0.8z^{-4}}{1 - 0.45z^{-1} + 0.7z^{-4}}$$

8.3 Design Techniques

The method used to calculate the filter coefficients (h_k for FIR, a_k and b_k for IIR) depends on whether the filter is IIR or FIR type.

There are several methods of calculating filter coefficients of which the following are the most widely used.

FIR digital filters	IIR digital filters
• window	• Impulse invariant transformation
• Frequency sampling	• Bilinear transformation
• Optimisation method (e.g Remez Algorithm)	• Pole-zero placement method
$\left\{ \frac{Y(z)}{X(z)} = h_0 + h_1 z^{-1} + h_2 z^{-2} + ... + h_{N-1} z^{-(N-1)} \right\}$	$\left\{ \frac{Y(z)}{X(z)} = \frac{a_0 + a_1 z^{-1} + ... + a_n z^{-(N-1)}}{1 + b_1 z^{-1} + ... + b_L z^{-L}} \right\}$

We choose the method that best suits our particular applications.

In most cases, if the FIR properties are vital then a good candidate is the optimization method, where as, if IIR properties are desirable, then the bilinear method will in most cases suffice.

8.4 IIR Filter Design

In transforming an analogue filter to digital filter, we must obtain either $H(z)$ or $h[n]$ from the analogue filter design. In such transformations, we generally require that the essential properties of the analogue frequency response be preserved in the frequency response of the resulting digital filter. This implies that we want the imaginary axis of the s-plane to map into the unit circle of z-plane.

A second condition is that a stable analogue filter should be transformed to a stable digital filter. That is if the analogue system has two poles only in the left half s-plane, then the digital filter must have poles inside the unit circle.

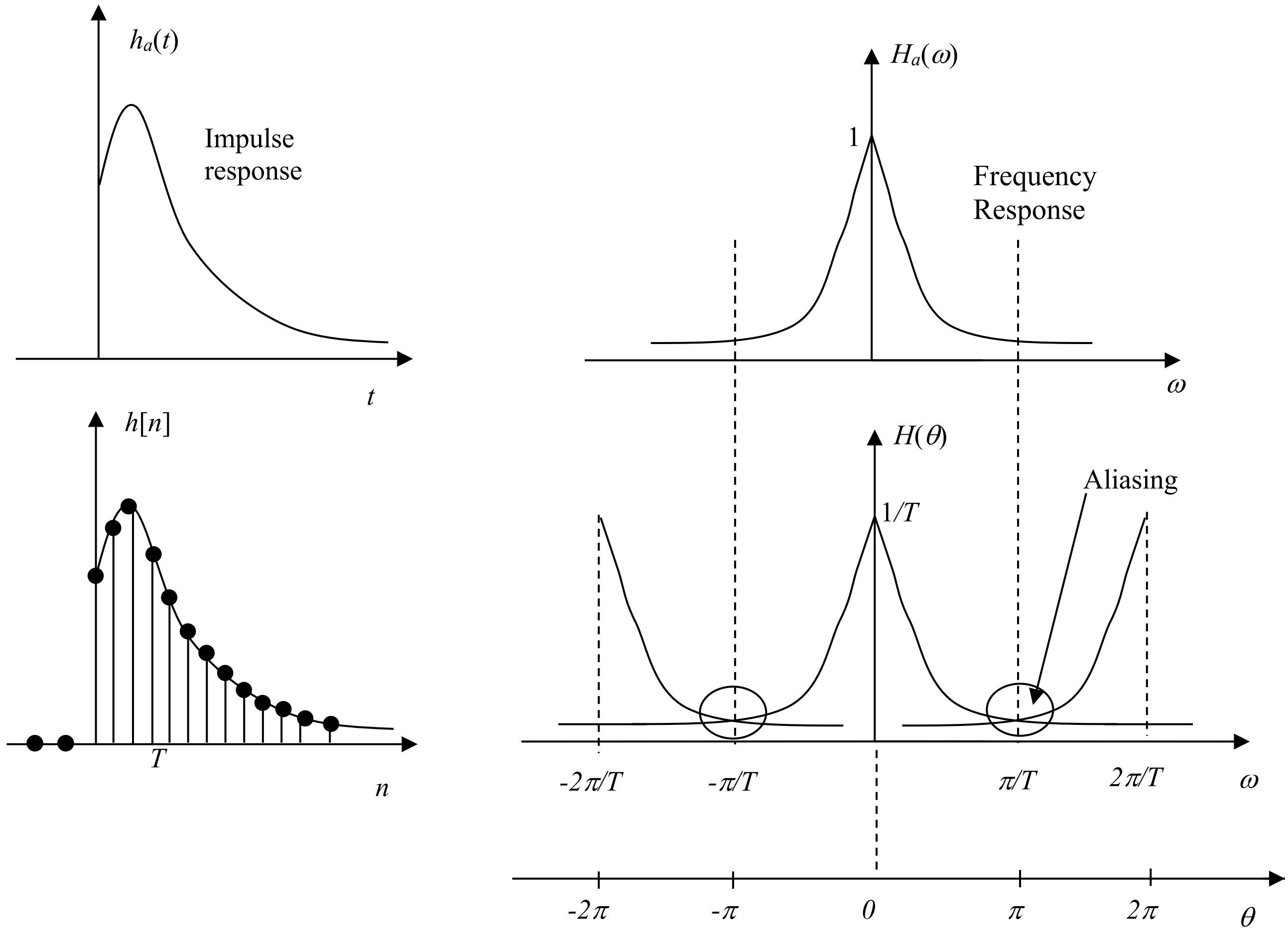
8.4.1 Impulse invariant method

In this method we start from an analogue filter of impulse response $h_a(t)$ and the system function $H_a(s)$.

The objective of our design is to realize an IIR filter with an impulse response $h[n]$ which satisfies :

$h[n] = h_a(nT)$ where T-sampling frequency

The characteristic property preserved by this transformation is that the impulse response of the resulting digital filter is a sampled version of the impulse response of the analogue filter.



We see that with this method there are problems to a greater or lesser extent depending on the choice of T.

The sampling frequency affects the frequency response of the impulse invariant discrete filter. A sufficient high sampling frequency is necessary for the frequency response to be close to that of the equivalent analogue filter. Thus due to aliasing, the frequency response of the digital filter will not be identical to that of the analogue filter. So how do we find the filter coefficients of the IIR filter in this design method?

To obtain the mapping let,

$$H_a(s) = \frac{1}{s+b}, \quad b > 0$$

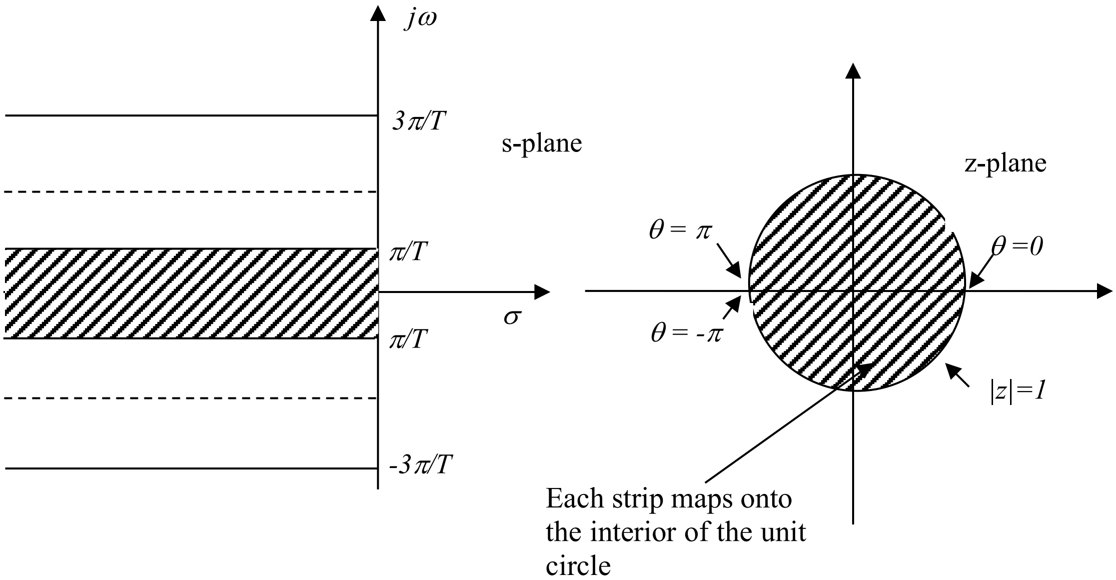
Inverse Laplace transform, $h_a(t) = e^{-bt}$

Sampled sequence, $h_a(nT) = e^{-bnT}$

Usually written as $h[n] = \begin{cases} e^{-bnT} & n \geq 0 \\ 0 & n < 0 \end{cases}$

z-transform, $H(z) = \dfrac{1}{1 - e^{-bT} z^{-1}}$

It is seen that $H(z)$ is obtained from $H_a(s)$ by using the mapping relationship, $\dfrac{1}{s+b} \to \dfrac{1}{1-e^{-bT} z^{-1}}, \quad b > 0,$ T-sampling period



In this kind of mapping, the perimeter is the imaginary axis.
Note: Mapping does not exist for the zeros.

Example:

$$H(s)=\frac{2}{(s+1)(s+3)}$$

Using partial fractions this can be written as

$$H(s)=\frac{1}{s+1}-\frac{1}{s+3}$$

$$\downarrow \quad \frac{1}{s+b}\rightarrow\frac{1}{1-e^{-bT}z^{-1}}$$

$$H(z)=\frac{1}{1-e^{-T}z^{-1}}-\frac{1}{1-e^{-3T}z^{-1}}$$
$$=\frac{\left(e^{-T}-e^{-3T}\right)z^{-1}}{1-(e^{-T}-e^{-3T})z^{-1}+e^{-4T}z^{-2}}$$

Example:

Use the impulse invariant method to design digital filter from an analogue prototype that has a system function.

$$H(s)=\frac{s+a}{(s+a)^2+b^2}$$

To design a filter using the impulse invariant method, expand *H(s)* in a partial fraction form:

$$H(s)=\frac{s+a}{\left[s+\left(a+jb\right)\right]\left[s+\left(a-jb\right)\right]}=\frac{A}{s+(a+jb)}+\frac{B}{s+(a-jb)}$$
$$=\frac{1/2}{s+(a+jb)}+\frac{1/2}{s+(a-jb)}$$
$$=\frac{1}{2}\left[\frac{1}{s+(a+jb)}\right]+\frac{1}{2}\left[\frac{1}{s+(a-jb)}\right]$$

Substituting $\frac{1}{s+c} \rightarrow \frac{1}{1-e^{-cT}z^{-1}}$ (Impulse invariant transform)

$$H(z) = \frac{1}{2}\left[\frac{1}{1-e^{-(a+jb)T}z^{-1}}\right] + \frac{1}{2}\left[\frac{1}{1-e^{-(a-jb)T}z^{-1}}\right]$$

$$= \frac{\frac{1}{2}\left[1-e^{-(a-jb)T}z^{-1}\right] + \frac{1}{2}\left[1-e^{-(a+jb)T}z^{-1}\right]}{(1-e^{-(a+jb)T}z^{-1})(1-e^{-(a-jb)T}z^{-1})}$$

$$= \frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}$$

Hence,

$$H(z) = \frac{1+a_1 z^{-1}}{1+b_1 z^{-1}+b_2 z^{-2}}$$

where

$$a_1 = e^{-aT}\cos(bT), \quad b_1 = -2e^{-aT}\cos(bT) \quad \text{and} \quad b_2 = e^{-2aT}$$

$$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}$$

Note that the zero at $s = -a$ is mapped to a zero at $z=e^{-aT}\cos(bT)$.

Thus, the location of the zero in the discrete time filter depends on the position of the poles as well as the zero in the analogue filter.

Example:

Using impulse invariant method design a digital filter to approximate the following normalized analogue transfer function: $H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$

Assume that the 3dB cut-off frequency of the digital filter is 150Hz and the sampling frequency is 1.28 kHz

Solution: Before applying the impulse invariant method, we need to de-normalize the transfer function.

$$H_1(s) = H(s)\Big|_{s \to \frac{s}{\omega_c}} = \frac{1}{\left(\frac{s}{\omega_s}\right)^2 + \sqrt{2}\left(\frac{s}{\omega_s}\right) + 1}$$

$$H(s) = \frac{\omega_c{}^2}{s^2 + \sqrt{2}\omega_c s + \omega_c{}^2} \quad \text{where} \quad \omega_c = 2\pi \times 150 = 942.4778$$

$$H_1(s) = \frac{\omega_c^2}{s + \sqrt{2}\omega_c s + \omega_c{}^2} = \frac{A}{s - p_1} + \frac{B}{s - p_2}$$

$$\begin{cases} p_1, p_2 = \frac{-\sqrt{2}\omega_c \pm \sqrt{2\omega_c^2 + 4\omega_c^2}}{2} \\ \quad = \frac{-\sqrt{2}\omega_c \pm j\sqrt{2}\omega_c}{2} \\ \quad = \frac{-\sqrt{2}\omega_c[1 \pm j]}{2} \\ \therefore p_1 = \frac{-\sqrt{2}\omega_c}{2}[1 + j] \\ p_2 = \frac{-\sqrt{2}\omega_c}{2}[1 - j] \end{cases}$$

$$\begin{cases} p_1 = -666.4324(1 + j) \\ p_2 = -666.4324(1 - j) \\ A = -\frac{\omega_c}{\sqrt{2}} j \\ B = \frac{+\omega_c}{\sqrt{2}} j \end{cases}$$

$$H_1(z) = \frac{A}{1 - e^{p_1T}z^{-1}} + \frac{B}{1 - e^{p_2T}z^{-1}} = \frac{A(1 - e^{-p_2T}z^{-1}) + B(1 - e^{p_1T}z^{-1})}{(1 - e^{p_1T}z^{-1})(1 - e^{p_2T}z^{-1})}$$

$$H(z) = \frac{(A + B) - (Ae^{p_2T} + Be^{p_1T})z^{-1}}{1 - (e^{p_1T} + e^{p_2T})z^{-1} + e^{p_1T}z^{-2}}$$

$$H(z) = \frac{393.9264z^{-1}}{1 - 1.0308z^{-1} + 0.3530z^{-2}}$$

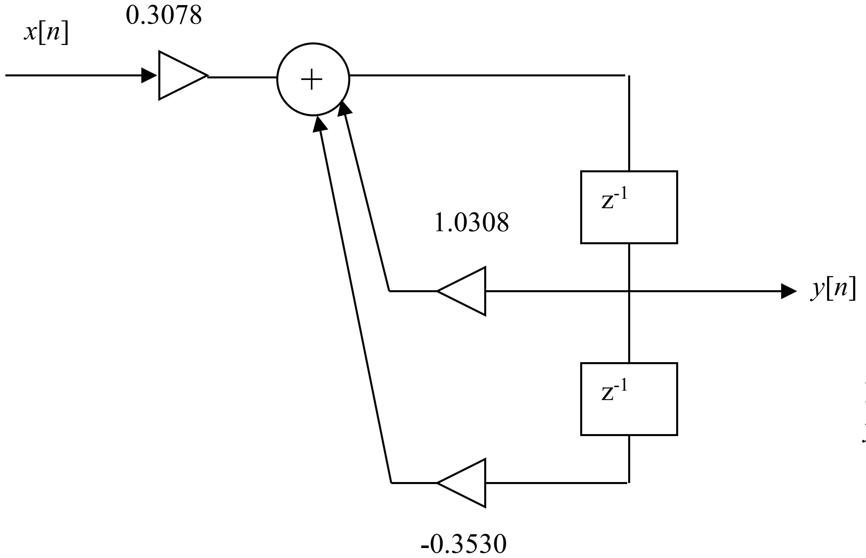
$$H(\theta) = \frac{393.9264e^{-j\theta}}{1 - 1.0308e^{-j\theta} + 0.3530e^{-j2\theta}}$$

$$H(\theta)\Big|_{\theta=0} = \frac{393.9264}{1 - 1.0308 + 0.3530} \approx 1223$$

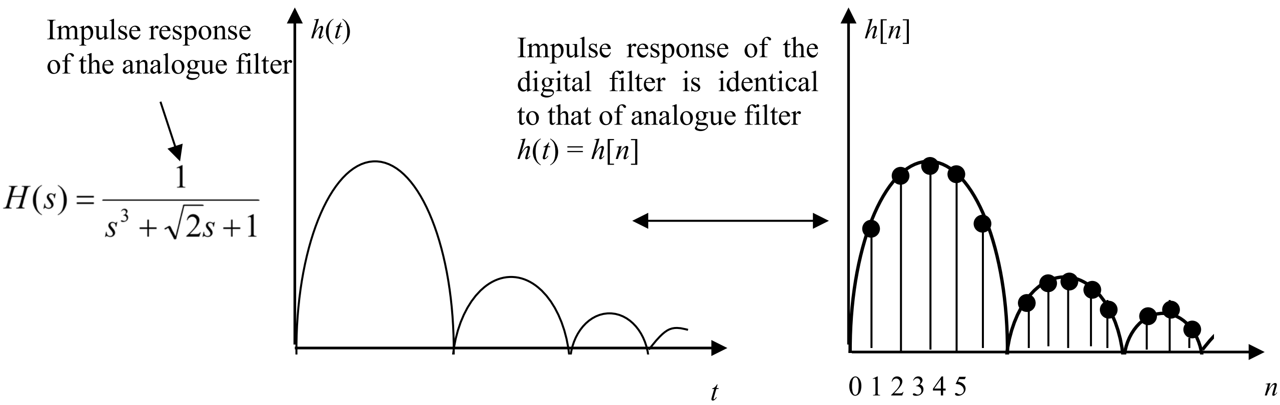
This is approximately equal to the sampling frequency

Such a large gain is characteristic of impulse invariant filters.To keep the gain down (and to avoid overflows when the filter is implemented). It is common practice to divide the gain by f_s. Thus the new transfer function becomes

$$H(z) = \frac{0.3078z^{-1}}{1 - 1.0308z^{-1} + 0.3530z^{-2}}$$

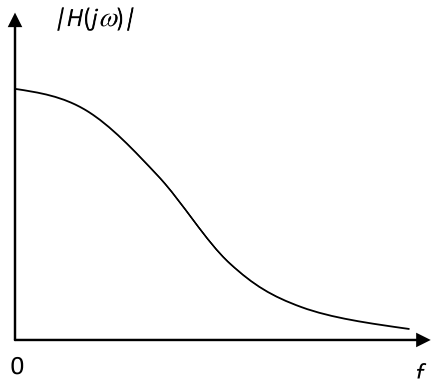


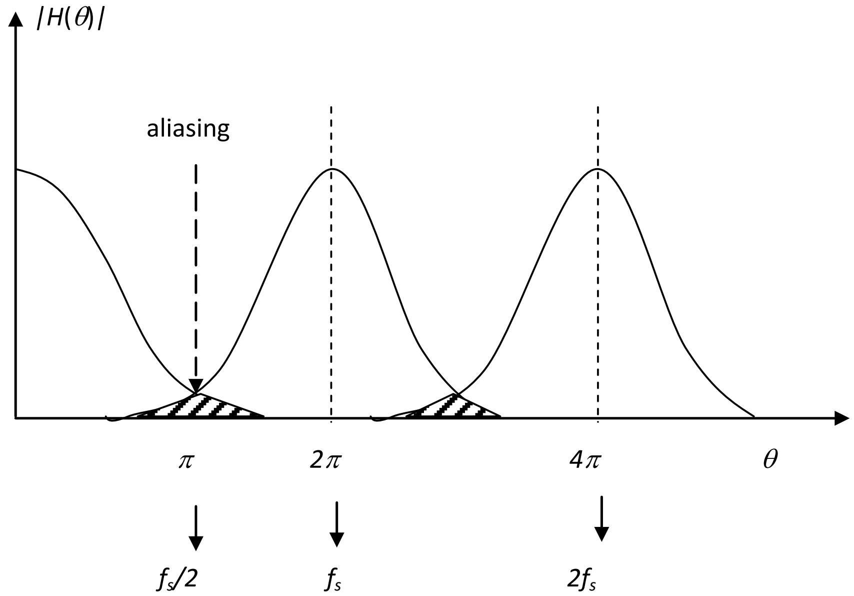
If $x[n]=\delta[n]$ then $y[n]=h[n]$.



Note: The sampling frequency affects the frequency response of the digital filter obtained using impulse invariant transformation.

A sufficient high samples frequency is necessary for the frequency response to be closer to that of the equivalent analogue filter (see below)

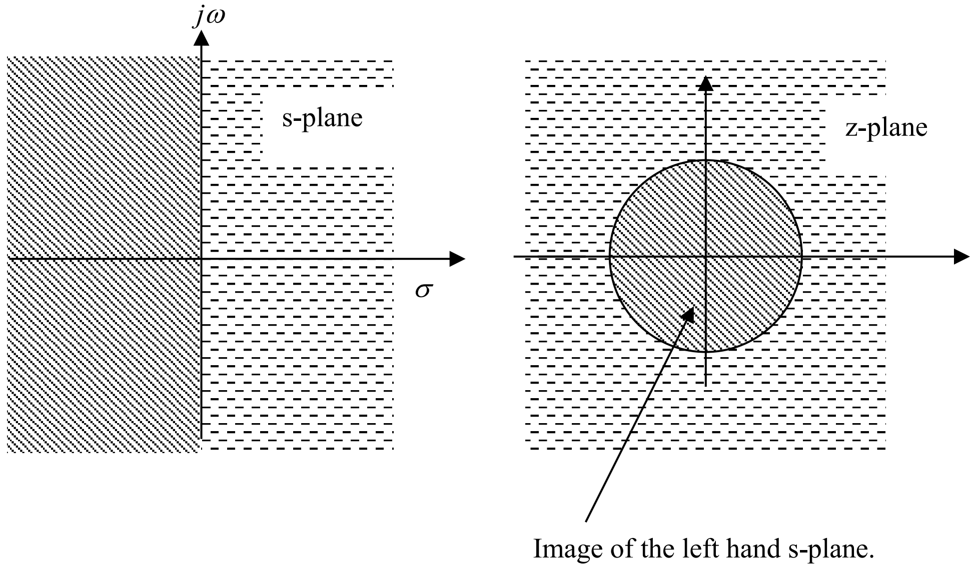




Low degree of aliasing can be achieved by making the sampling frequency high.

8.4.2 Bilinear Transformation

The bilinear transformation yields stable digital filters from stable analogue filters (the impulse invariant technique may not). Also the bilinear transformation avoids the problem of aliasing encountered with the use of the impulse invariant transformation, because it maps the entire imaginary axis in the s-plane on to the unit circle in the z-plane.



$$z = e^{sT} = \frac{e^{\frac{sT}{2}}}{e^{\frac{-sT}{2}}} = \frac{1 + \frac{sT}{2} + \left(\frac{sT}{2}\right)^2 \cdot \frac{1}{2} + \ldots}{1 - \frac{sT}{2} + \left(\frac{sT}{2}\right)^2 \cdot \frac{1}{2} + \ldots}$$

$$z \approx \frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}}$$

(drop out higher order terms)

$$\therefore s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$

The price paid for the avoidance of aliasing is an introduction of distortion in the frequency axis. Consequently, the design of digital filters using the bilinear transformation is only useful when the distortion can be compensated.

Example: We have a system function $H_a(s)$ such that

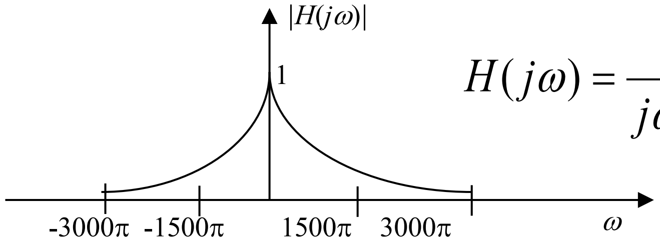
$$H_a(s) = \frac{b}{s+b}$$

Applying bilinear transformation,

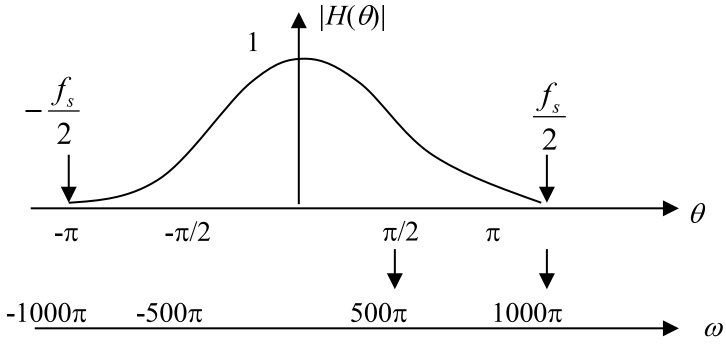
$$H(z) = \frac{b}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + b} = \frac{bT(1+z^{-1})}{bT + 2 + (bT-2)z^{-1}}$$

Let b = 1000 & $T = \dfrac{1}{1000}$

The modulus of the frequency response $H(\omega)$ and $H(\theta)$ are shown below:



$$H(j\omega) = \frac{b}{j\omega + b}$$



There is a very important property of the bilinear transformation that can be seen in the above example. The entire frequency range ($-\infty \le \omega_a \le \infty$) of the continuous system maps into the fundamental interval ($-\pi \le \theta \le \pi$) of the discrete system, where $\omega = 0$ corresponds to $\theta = 0$, $\omega = \infty$ to $\theta = \pi$ and $\omega = -\infty$ to $\theta = -\pi$.

To demonstrate that this mapping has the property that the imaginary axis in the s-plane maps onto the unit circle, let $z = e^{j\theta}$ and s = $j\omega$. Then since $s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$,

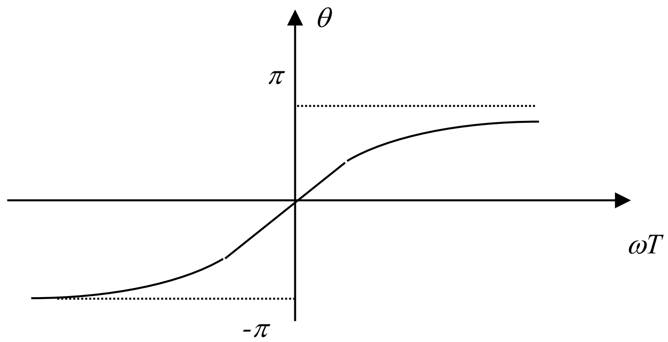
$$j\omega = \frac{2}{T}\frac{1 - e^{-j\theta}}{1 + e^{-j\theta}}$$

$$j\omega = \frac{2}{T} j \tan\left(\frac{\theta}{2}\right)$$

Hence,

$$\omega = \frac{2}{T}\tan\left(\frac{\theta}{2}\right) \text{ or } \theta = 2\tan^{-1}\left(\frac{\omega T}{2}\right)$$

We see that a nonlinear relation exists between ω and θ. This effect is called 'Warping' and is shown below.



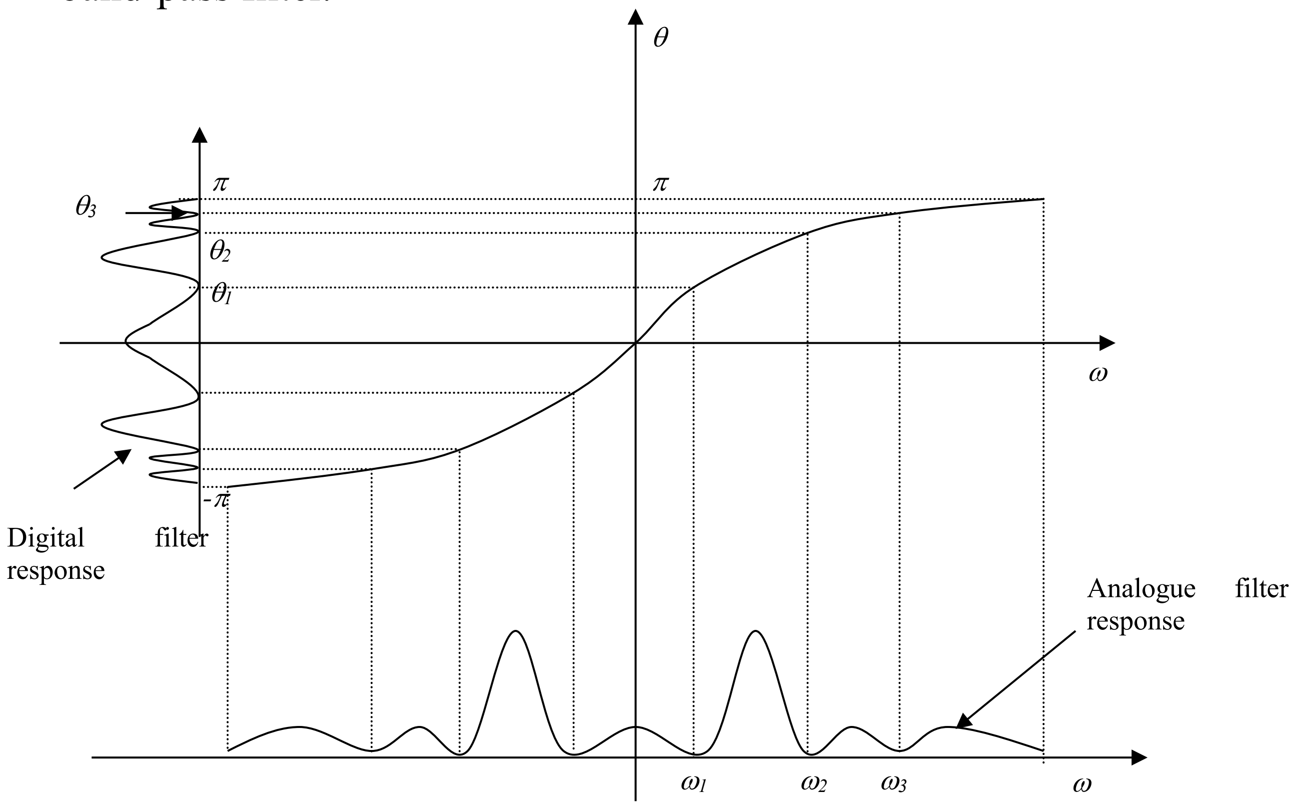
At low frequencies $\tan^{-1}\left(\frac{\omega T}{2}\right) = \frac{\omega T}{2}$.

$$\therefore \theta = 2 \times \frac{\omega T}{2} = \omega T$$

The great advantage of warping is that no aliasing of the frequency characteristic can occur in the transformation of an analogue filter to a discrete filter, which we encountered in the impulse-invariant method.

We must however check carefully just how the various characteristic frequencies of the continuous characteristic frequencies of the discrete filter.

We can illustrate this with the aid of a diagram (below) for a band-pass filter.



The effect of "warping" in the conversion of $|H(\omega)| \rightarrow |H(\theta)|$ is seen from the above diagram.

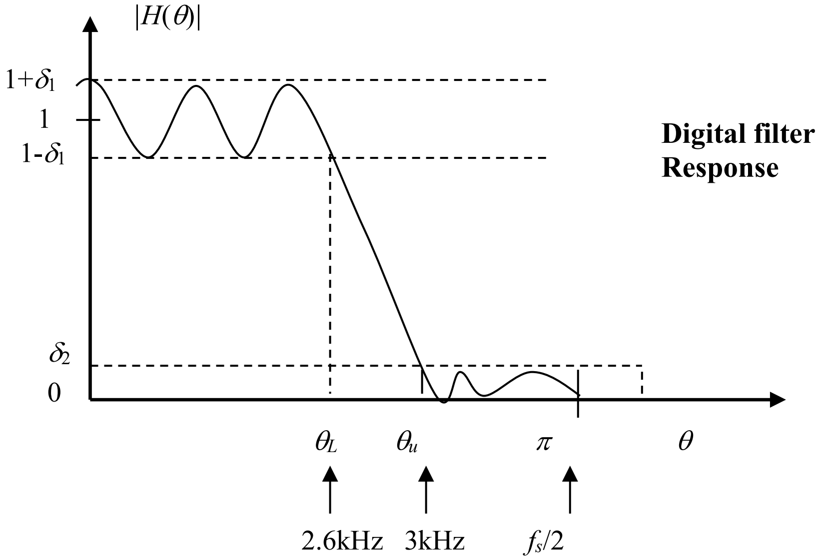
In designing a digital filter by this method we must first **pre-wrap** the given filter specifications to find the continuous filter to which we are going to apply the bilinear transformation.

Example: The specification of a desired digital low pass filter is shown below.

Sampling frequency: f_s = 8kHz ($T = 1/f_s$ = 125μs)

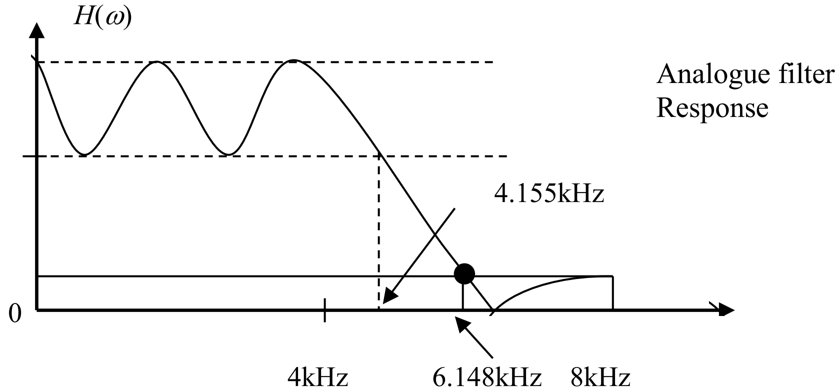
A pass band up to f_L = 2.6 kHz ($\theta_L = \omega_L T = 2\pi \dfrac{f_L}{f_s} = 0.65\pi$)

and a stop band, f_u, above 3kHz ($\theta_u = \omega_L T = 2\pi \dfrac{f_u}{f_s} = 0.75\pi$).



We must start from an analogue filter with

$$\omega_L = 2\pi f_l = \frac{2}{T}\tan\frac{\theta_L}{2} = 2\pi(4155) \Rightarrow f_L = 4.155\text{kHz}$$
$$\omega_u = 2\pi f_h = \frac{2}{T}\tan(\frac{\theta_u}{2}) = 2\pi(6148) \Rightarrow f_u = 6.148\text{kHz}$$



Example: Determine, using bilinear transformation method, the transfer function and difference equation for the digital equivalent of the RC filter. The normalized transfer function for the RC filter is

$$H(s) = \frac{1}{s+1}$$

Assume a sampling frequency of 150Hz and a cut-off frequency of 30Hz.

Solution: $\theta_c = \omega_c T = 2\pi f_c \cdot \frac{1}{f_s}$ before prewarping

$$\theta_c = 2\pi(30) \cdot \frac{1}{150} = 0.4\pi$$

The analogue frequency after prewarping is

$$\omega_c' = \frac{2}{T}\tan\left(\frac{0.4\pi}{2}\right) = 217.95 \text{ rad/sec}$$

Pre-warped frequency

f_c' is always > f_c (un-warped freq= 30 Hz) and after warping, f_c' = 34.68Hz.

The de-normalized analogue filter transfer function (this is achieved by replacing s with $\frac{s}{\omega_c}$) is obtained from $H(s)$ as

$$H_1(s) = H(s)\Big|_{s=\frac{s}{\omega'_c}} = \frac{\omega_c'}{s+\omega_s'} = \frac{\left(\frac{2}{T}\right)0.7265}{s+\frac{2}{T}\times 0.7265}$$

$$H(z) = H_1(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}} \cdot = \frac{0.7265(1+z)}{(1+7265)z+0.7265-1} = \frac{0.4208(1+z^{-1})}{1-0.1584z^{-1}}$$

Example:

It is required to design a digital filter to approximate the following analogue transfer function

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

using the Bi-linear transformation method obtain the transfer function, *H(s)* of the digital filter assuming a 3dB cut-off frequency of 150Hz and a sampling frequency of 1.28kHz.

Solution:

$$f_c = 150\text{Hz} \qquad f_s = 1280\text{Hz} \quad T = \frac{1}{f_s}$$

$$\downarrow$$

$$\theta_c = 2\pi\left(\frac{f_c}{f_s}\right) = 2\pi\left(\frac{150}{1280}\right) = \frac{15}{64}\pi$$

The analogue frequency after pre-warping

$$\omega_c' = \frac{2}{T}\tan\left(\frac{\theta_c}{2}\right) = \frac{2}{T}\times 1280\tan\left(\frac{\frac{15\pi}{64}}{2}\right) = 987.5009 = \frac{2}{T}\times 0.3859 rad/\sec$$

$$f_c' = \frac{987.5009}{2\pi} = 157.1656Hz$$

f_c' is always greater than f_c.

Prewarped analogue filter is given by,

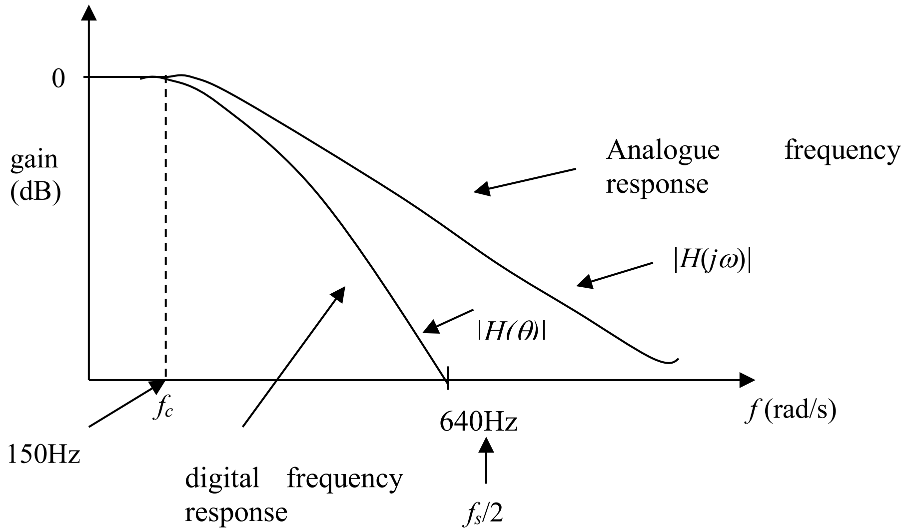
$$H_1(s) = \frac{\left(\frac{2}{T}\right)^2 (0.3857)^2}{s^2 + \frac{2}{T}(0.3857)\sqrt{2}s + \left(\frac{2}{T}\right)^2 (0.3857)^2}$$

$$H(z) = H_1(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}} = \frac{\left(\frac{2}{T}\right)^2 (0.3857)^2}{\left(\frac{2}{T}\frac{z-1}{z+1}\right)^2 + \frac{2}{T}\times 0.3857\times\sqrt{2}\left(\frac{2}{T}\frac{z-1}{z+1}\right) + \left(\frac{2}{T}\right)^2 (0.3857)^2}$$

$$H(z) = \frac{0.0878\left(1 + 2z^{-1} + z^{-2}\right)}{1 - 1.0048z^{-1} + 0.3561z^{-2}}$$

$$\frac{\left(\omega_c'\right)^2}{s^2 + \sqrt{2}\omega_c' s + \left(\omega_{c'}\right)^2} \xrightarrow{\textit{Bilinear transformation}} \frac{0.0878(1 - 2z^{-1} + z^{-2})}{1 - 1.0048z^{-1} + 0.3561z^{-2}}$$

All pole   Poles & zeros



Note:

- Same cut-off frequency
- increased roll off and attenuation in stop band.

Example:

(a) An analogue transfer function can be converted to a digital transformation using the bilinear transformation. Derive this transform relationship using the following equation.

$$y[n]-y[n-1]=\frac{T}{2}\left[x[n]+x[n-1]\right] \longleftarrow \text{Digital Integrator}$$

$$H(s)=\frac{1}{s} \longleftarrow \text{Analogue Integrator}$$

T-sampling period, $x[n]$ - input , $y[n]$ - output

Solution:

$$Y(z)-Y(z)z^{-1}=\frac{T}{2}\left\{X(z)-X(z)z^{-1}\right\}$$

$$H(z)=\frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} \qquad (1)$$

$$H(s)=\frac{1}{s} \qquad (2)$$

Setting (1) = (2),

$$\frac{1}{s}=\frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} \quad \Rightarrow s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

(b) Convert the analogue filter $H(s)$

$$H(s)=\frac{s+0.1}{(s+0.1)^2+16}$$

into a digital IIR filter by means of the bilinear transformation.

The digital filter is to have a resonant frequency $\theta_0 = \frac{\pi}{2}$.

$$H(s) = \frac{s + 0.1}{s^2 + 0.2s + 16.01}$$

← $\omega_0{}^2$

The analogue filter has a resonant frequency $\omega_0 \approx 4$ rad/sec.

The frequency is to be mapped into $\theta_0 = \frac{\pi}{2}$ by selecting the value of the parameter T.

$$\omega_0 = \frac{2}{T}\tan\frac{\theta_0}{2}$$

$$4 = \frac{2}{T}\tan\frac{\frac{\pi}{2}}{2} = \frac{2}{T}\tan\frac{\pi}{4} \quad \Rightarrow 4 = \frac{2}{T}$$

$\therefore T = \frac{1}{2}$ in order to have $\theta_0 = \frac{\pi}{2}$ rad/sec

Thus the desired mapping is

$$s = 4\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

$$H(z) = H(s)\Big|_{s=4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{4\left(\frac{1 - z^{-1}}{1 - z^{-1}}\right) + 0.1}{4\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)^2 + 0.2 \times 4\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + 16.01}$$

$$= \frac{0.128 + 0.006z^{-1} - 0.122z^{-2}}{1 + 0.0006z^{-1} + 0.975z^{-2}} = \frac{0.128 + 0.006z^{-1} - 0.122z^{-2}}{1 + 0.975z^{-2}}$$

This filter has poles at

$$p_1 = 0.987e^{+j\frac{\pi}{2}}$$

$$p_2 = 0.987e^{-j\frac{\pi}{2}}$$

← resonant frequency

and zeros at $z_1 = -1$ and $z_2 = 0.95$.

Example: Convert the simple low pass filter $H(s)=\dfrac{1}{s+1}$ into an equivalent high pass discrete filter. Assume f_s = 150Hz, analogue cut-off f_c = 30Hz.

$$\theta_c = 2\pi\left(\frac{30}{150}\right) = 0.4\pi$$

$$\omega_c^{'} = \frac{2}{T}\tan\left(\frac{0.4\pi}{2}\right) = \frac{2}{T}\times 0.7265$$

LPF to HPF transformation

$$H^{'}(s) = H(s)\Big|_{s\to\frac{\omega_c^{'}}{s}} = \frac{1}{\frac{\omega_c^{'}}{s}+1} = \frac{s}{s+\omega_c^{'}}$$

$$H(z) = H^{'}(s)\Big|_{z=\frac{2}{T}\left(\frac{z-1}{z+1}\right)} = \frac{\frac{2}{T}\left(\frac{z-1}{z+1}\right)}{\frac{2}{T}\left(\frac{z-1}{z+1}\right)+\frac{2}{T}0.7265} = \frac{z-1}{z-1+0.7265(z+1)}$$

$$H(z) = 0.5792\frac{1-z^{-1}}{1+0.1584z^{-1}}$$

Exercise:

Using the second order normalised Butterworth lowpass filter G(s), design an IIR digital filter G(z) to approximate the analogue transfer function G(s). The digital filter is to be designed to operate at a sampling rate of 8 kHz with a 3 dB cut-off frequency at 2 kHz. Using the bilinear transformation method, obtain the transfer function G(z) of the digital filter and show that G(z)

$$G(z) = \frac{(z+1)^2}{(2+\sqrt{2})(z^2+\frac{2-\sqrt{2}}{2+\sqrt{2}})}$$

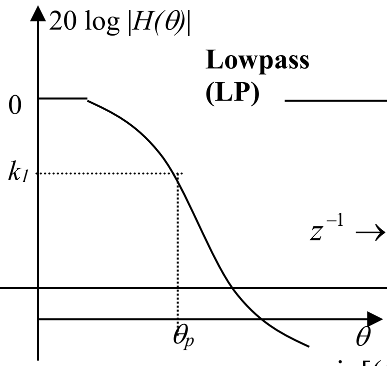
8.4.3 Digital-To-Digital Transformations

We have seen in the lecture notes that one method for the design of analogue filters relied on applying a transformation to an analogue low-pass filter with a unit bandwidth. It was shown that we could obtain low-pass, high-pass, band-pass and band-stop filters by selecting the appropriate transformation.

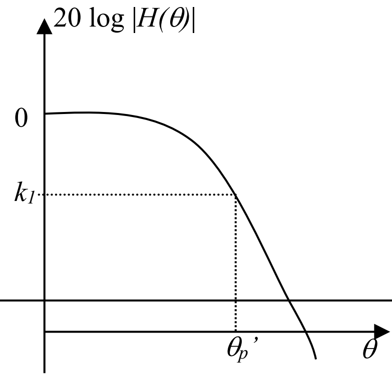
Similarly, a set of transformations can be formed that take a low-pass digital filter and turn it into high-pass, band-pass, and band-stop or another low-pass digital filter.

The transformations are given below:

Digital to Digital Transformations



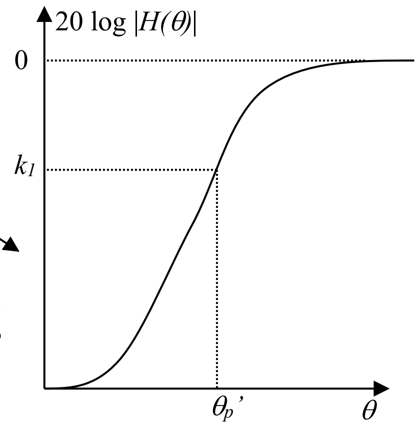
Lowpass (LP) → **Lowpass (LP)**



$$z^{-1} \rightarrow \frac{z^{-1} - \alpha}{1 - \alpha\, z^{-1}}$$

$$\alpha = \frac{\sin[(\theta_p - \theta_p')/2]}{\sin[(\theta_p + \theta_p')/2]}$$

Lowpass to Highpass

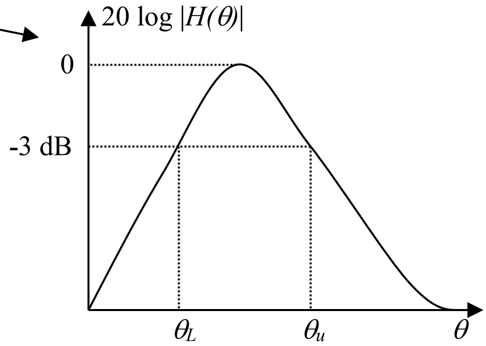


$$z^{-1} \rightarrow \frac{z^{-1} + \alpha}{1 + \alpha\, z^{-1}}$$

Simplest transformation is to change the signs of poles and zeros of the LP → LP transformation

Lowpass to Bandpass

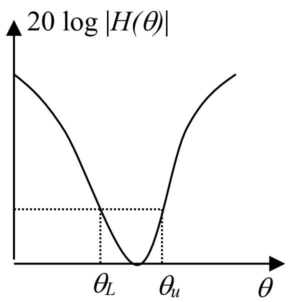
$$\alpha = \frac{\cos[(\theta_p + \theta_p')/2]}{\cos[(\theta_p - \theta_p')/2]}$$



$$z^{-1} \rightarrow -\frac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$$

$$\alpha = \frac{\cos[(\theta_u + \theta_L)/2]}{\cos[(\theta_u - \theta_L)/2]}$$

$$k = \cot((\theta_u - \theta_L)/2) \cdot \tan\left(\frac{\theta_p}{2}\right)$$



Lowpass to Bandstop

$$z^{-1} = \frac{z^{-2} - \frac{2\alpha}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha}{k+1} z^{-1} + 1}$$

8.5 Notch filters

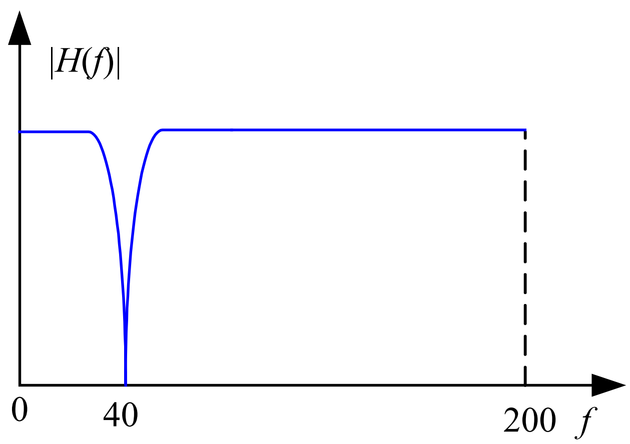
When a zero is placed at a given point on the z-plane, the frequency response will be zero at the corresponding point. A pole on the other hand produces a peak at the corresponding frequency point.

Poles that are close to the unit circle give rise large peaks, where as zeros close to or on the unit circle produces troughs or minima. Thus, by strategically placing poles and zeros on the z-plane, we can obtain sample low pass or other frequency selective filters (**notch filters**).

<u>Example:</u>
Obtain, by the pole-zero placement method, the transfer function of a sample digital notch filter (see figure below) that meets the following specifications:

- Notch Frequency: 40Hz
- 3db width of the Notch: ±4Hz
- Sampling frequency: 400 Hz

The radius, r of the poles is determined by: $r = 1 - \left(\frac{\Delta f}{f_s}\right)\pi$

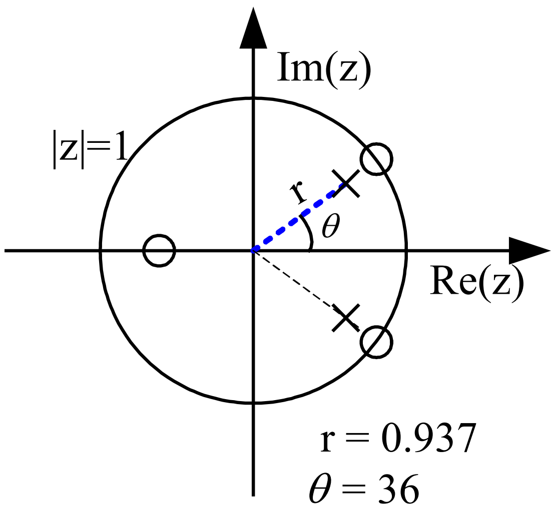


To reject the component at 40 Hz, place a pair of complex zeros at the points on the unit circle corresponding to ±40 Hz.

$$\text{i.e. at angles of } 360^{\bullet} \times \frac{40}{400} = \pm 36^{\bullet} = \pm 0.2\pi$$

To achieve a sharp notch filter and improved magnitude response either side of the notch frequency, a pair of complex conjugate poles are placed at a radius $r < 1$, at the same frequency as the zeroes, as shown below.

$$r = 1 - \left(\frac{\Delta f}{f_s}\right)\pi = 1 - \left(\frac{8}{400}\right)\pi = 0.937$$



$$H(z) = \frac{\left(z - e^{-j0.2\pi}\right)\left(z - e^{j0.2\pi}\right)}{\left(z - 0.937e^{-j0.2\pi}\right)\left(z - 0.937e^{j0.2\pi}\right)}$$

$$= \frac{z^2 + 1 - (e^{j0.2\pi} + e^{-j0.2\pi})}{z^2 + 0.878 - 0.937(e^{j0.2\pi} + e^{-j0.2\pi})z}$$

$$= \frac{z^2 + 1 - 2\cos(0.2\pi)}{z^2 + 0.878 - 2 \times 0.937\cos(0.2\pi)}$$

$$= \frac{1 - 1.6180z^{-1} + z^{-2}}{1 - 1.5161z^{-1} + 0.878z^{-2}}$$

Exercise: A bandpass digital filter is required to meet the following specifications:

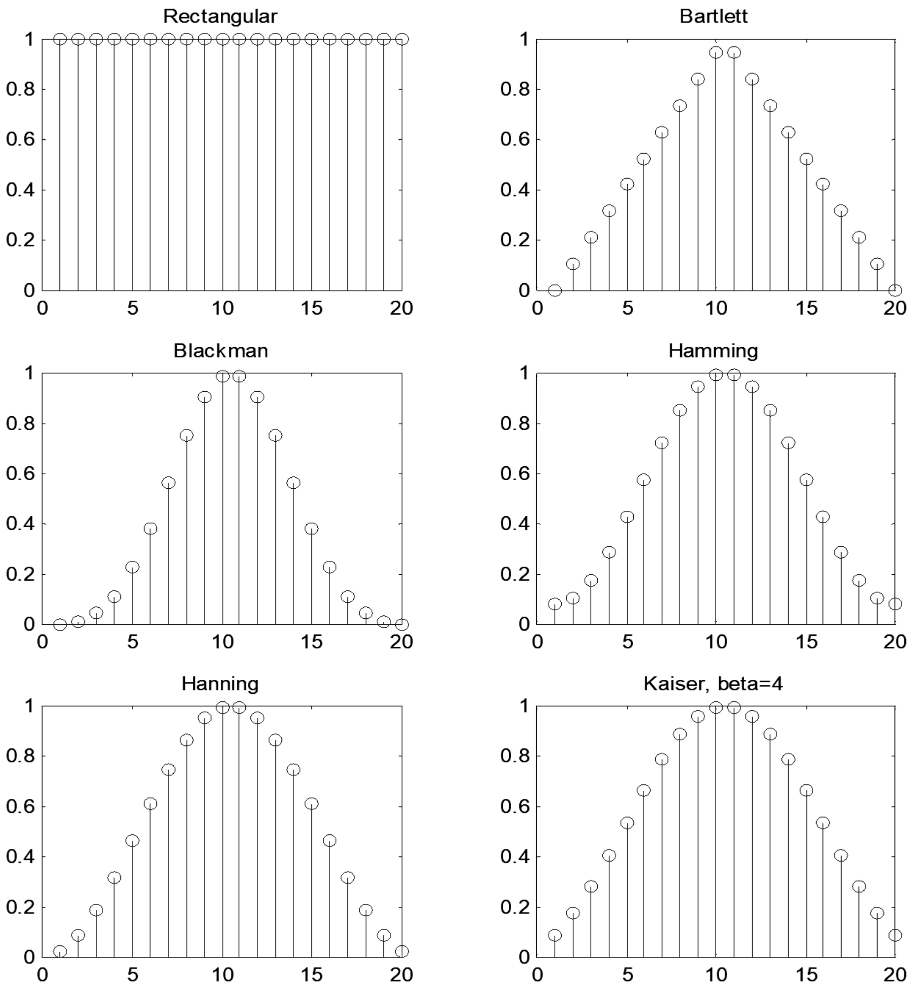
(1) Complete signal rejection at dc and 250Hz;
(2) A narrow passband centred at 125Hz;
(3) A 3dB bandwidth of 10Hz

Assuming a sampling frequency (f_s) of 500Hz, obtain the transfer function of the filter, by suitably placing z-plane poles and zeros, and it difference equations.

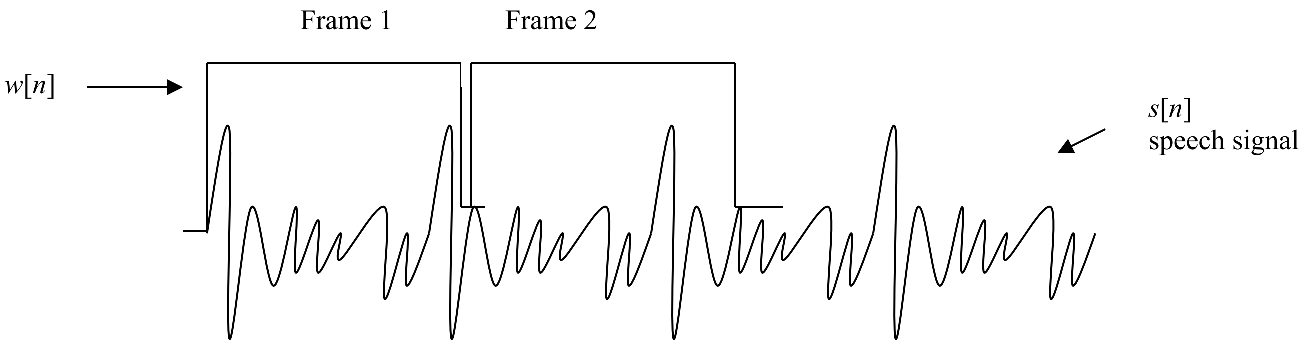
The radius, r of the poles is determined by the desired bandwidth. An approximate relationship between r, for r>0.9, and bandwidth, BW, is given by $r \approx 1 - \left(\frac{BW}{f_s}\right)\pi$

8.6 Window Functions

Window functions are used to truncate a signal or impulse response to produce a signal or impulse response of finite duration. Some of the most commonly used window functions are:



To analyse a **truncation** process we model it as a multiplication of the desired sequence by finite duration window sequence denote by $w[n]$. Truncation of a sequence $s[n]$ is equivalent to placing a rectangular time window around $s[n]$.



Vocal tract shape changes every 15ms. When using a sampling frequency of 8 kHz (T = 125μs) with 100 samples in each frame (12.5ms),

$$y[n] = s[n] \cdot w[n]$$

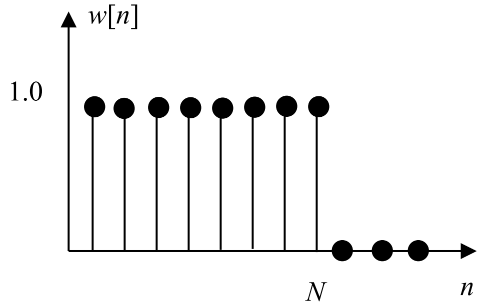
In frequency domain,

(Multiplication becomes convolution in the frequency domain)

$$Y(\theta) = S(\theta) * W(\theta)$$

Thus when window is applied, the frequency domain convolution causes distortion in the spectrum $Y(\theta)$. It can be shown that the rectangular window creates ripples in the spectrum $Y(\theta)$. To reduce the distortion, use Hamming or Kaiser Window.

Rectangular Window

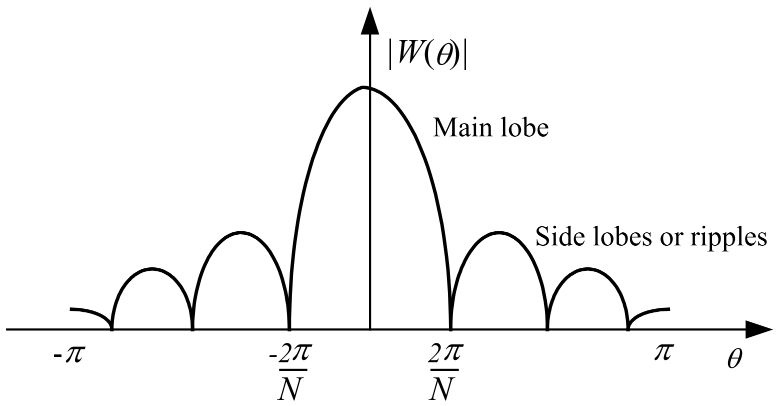


$$w[n]=\begin{cases}1 & 0\le n\le N-1\\ 0 & \text{otherwise}\end{cases}$$

$$W(z)=\sum_{n=0}^{N-1} w(n)z^{-n} = 1+z^{-1}+z^{-2}+.......+z^{-(N-1)} = \frac{1-z^{-N}}{1-z^{-1}}$$

$$W(\theta)=W(z)\Big|_{z=e^{j\theta}} = \frac{1-e^{-jN\theta}}{1-e^{-j\theta}}$$

$W(\theta)=e^{-j\frac{N-1}{2}\theta}\dfrac{\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}}$	$\phi(\theta)=-\left(\dfrac{N-1}{2}\right)\theta$	$\lvert W(\theta)\rvert=\left\lvert\dfrac{\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}}\right\rvert$



If *N* increases the width of the main lobe decreases but the peak amplitude of the side lobes grows in a manner such that the area under each lobe is constant while the width of each lobe decreases with *N*.

8.7 Design Methods for FIR filters

The most essential feature of FIR filters is, by definition, the finite length of the impulse response. Another important point is that it can be seem directly from the impulse response of an FIR filter whether we have a linear phase characteristic or not.

A filter is said to have a linear phase response of its response satisfies one of the following relationships.

$$\phi(\theta) = -a\theta \qquad (1)$$

$$\phi(\theta) = b - a\theta \qquad (2)$$

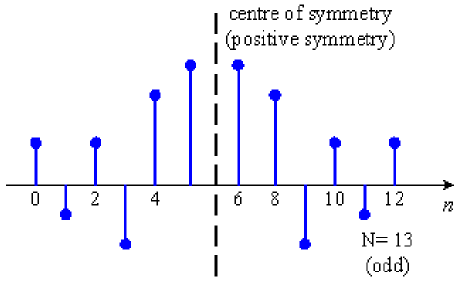
where a and b are constants.

It can be shown that for condition (1) above to be satisfied the impulse response of the filter must have positive symmetry.

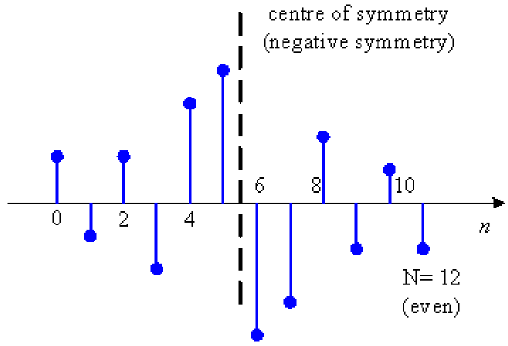
$$h[n] = h[N-n-1], \qquad a = \frac{N-1}{2}$$

where N denotes the filter length. When the condition given in (2) is satisfied, the impulse response of the filter has negative symmetry.

$$h[n] = -h[N-n-1], \quad a = \frac{N-1}{2} \text{ and } b = \frac{\pi}{2}$$



(a)



(b)

8.7.1 Design of FIR filters using Windows.

The easiest way to obtain an FIR filter is to simply truncate the impulse response of an IIR filter. If $h_d[n]$ represents the impulse response of a desired IIR filter, then an FIR filter with impulse response $h[n]$ can be obtained as follows:

$$h[n] = \begin{cases} h_d[n] & N_1 \le n \le N_2 \\ 0 & \text{otherwise} \end{cases}$$

In general $h[n]$ can be thought of as being formed by the product $h_d[n]$ and a window function $w[n]$ as follows:

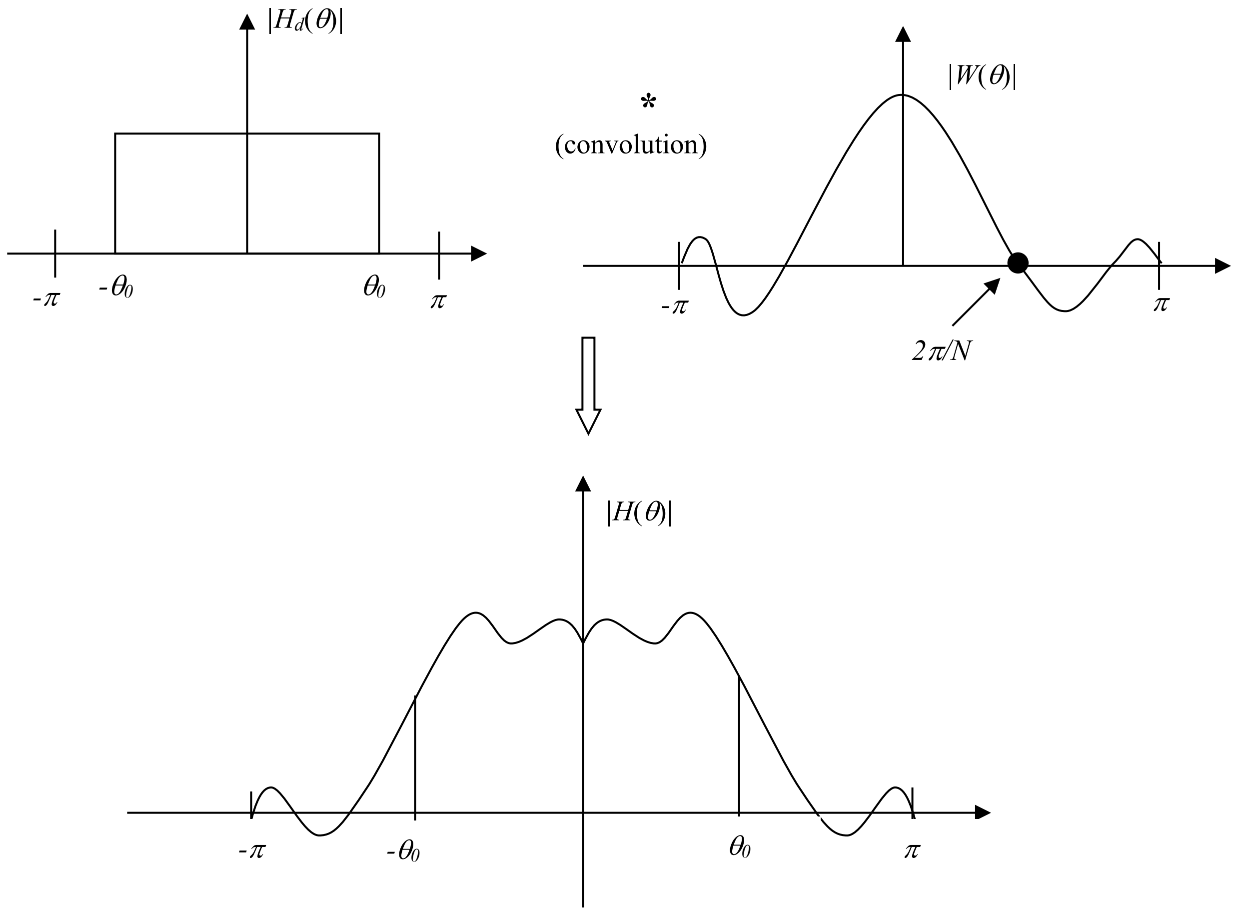
$$h[n] = h_d[n] \cdot w[n]$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$H(\theta) = H_d(\theta) * W(\theta)$$

$w[n]$: let it be, for example a rectangular window

$H_d(\theta)$: let it be an ideal low pass filter with cut off frequency θ_0



Therefore it is seen that the convolution produces a smeared version of the ideal low pass frequency response $H_d(\theta)$.

In general, the wider the main lobe of $W(\theta)$, the more spreading, where as the narrower the main lobe (larger N) the closer $|H(\theta)|$ comes to $|H_d(\theta)|$.

In general we are left with a trade off of making N large enough so that smearing is minimized, yet small enough to allow reasonable implementation.

8.7.2 Design Procedure

An ideal low pass filter with linear phase of slope $-\beta$ and cut-off ω_c can be characterized in the frequency domain by

$$H_d(\theta) = \begin{cases} e^{-j\theta\beta} & |w| \le \omega_c \\ 0 & w_c < |\omega| \le \pi \end{cases}$$

The corresponding impulse response $h_d[n]$ can be obtained by taking the inverse Fourier transform of $\boldsymbol{H}_d(\theta)$ and easily shown to be

$$h_d[n] = \frac{\sin[\omega_c(n-\beta)]}{\pi(n-\beta)}$$

A causal FIR filter with impulse response $h[n]$ can be obtained by multiplying $h_d[n]$ by a window beginning at the origin and ending at N-1 as follows :

$$h[n] = \frac{\sin[\omega_c(n-\beta)]}{\pi(n-\beta)}\omega[n]$$

For $h[n]$ to be a linear phase, β must be selected so that the resulting $h[n]$ is symmetric.

As $\dfrac{\sin[\omega_c(n-\beta)]}{\pi(n-\beta)}$ is symmetric about $n=\beta$ and the window symmetric is about $n=\dfrac{N-1}{2}$

$$\therefore \qquad \beta = \frac{N-1}{2}$$ ⟵ Symmetric about β

Example:

(a) Determine the impulse response $h_d[n]$ of the lowpass filter whose frequency response is given by

$$H(\theta) = \begin{cases} 1 & 0 \le |\theta| \le \frac{\pi}{3} \\ 0 & \frac{\pi}{3} < |\theta| \le \pi \end{cases}$$

(b) To obtain a finite impulse response from $h_d[n]$ a rectangular window of length $N = 9$ is used. Compute the coefficients of the FIR filter with a linear phase characteristic ands with this finite impulse response.

Solution:

(a)

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\theta)e^{j\theta n}d\theta$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\theta n}}{jn}\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = \frac{1}{\pi}\left(\frac{\sin n\frac{\pi}{3}}{n}\right)$$

$$h_d[n] = \begin{cases} 1/3 & n = 0 \\ \dfrac{\sin(n\pi/3)}{n\pi} & n \ne 0 \end{cases}$$

(b) For a linear phase filter, $h_d[n]$ is symmetric about n = 0 and the window is symmetric about $n = \frac{(N-1)}{2} = \frac{9-1}{2} = 4$.

$$h[n] = h_d[n] \cdot \omega[n]$$

$$h[0] = 0.333$$

$$h[1] = \frac{\sqrt{3}}{2\pi}$$

$$h[2] = \frac{\sqrt{3}}{4\pi}$$

$$h[3] = 0$$

$$h[4] = -\frac{\sqrt{3}}{8\pi}$$

The coefficients are

	$\frac{-\sqrt{3}}{8\pi}$	0	$\frac{\sqrt{3}}{4\pi}$	$\frac{\sqrt{3}}{2\pi}$	0.333	$\frac{\sqrt{3}}{2\pi}$	$\frac{\sqrt{3}}{4\pi}$	0	$\frac{-\sqrt{3}}{8\pi}$
n	0	1	2	3	4	5	6	7	8

Symmetry about n = 4

Exercise: The filter $G(\theta)$ need not be very sharp in practice. It only serves for attenuating higher spectral components resulting from the increase in the sampling rate. You are requested to design a linear phase FIR filter for this purpose. The length of the impulse response is set to N=7 and the coefficients are determined using a windowing technique. To obtain a linear phase FIR filter after windowing a phase factor, $f(\theta)$ must be first introduced.

$$G_d(\theta) = \begin{cases} e^{j\phi(\theta)} & 0 \le |\theta| \le \pi/4 \\ 0 & \pi/4 \le |\theta| \le \pi \end{cases}$$

(i) Determine $\phi(\theta)$.

(ii) Using part (i) above, Determine the impulse response $g_d[n]$ using a rectangular window.

CHAPTER 8: PROBLEM SHEET 8

Q1)

a) A second-order analogue band pass filter with an s-domain transfer function is given by

$$H(s) = \frac{b_p s}{s^2 + b_p s + \omega_p^2} \qquad (1)$$

Where ω_p and b_p are the centre frequency and bandwidth of the filter; respectively, both expressed in rad/s. By applying the bilinear transformation to equation (1) a digital filter with the following transfer function can be obtained:

$$H(z) = \frac{a_0 - a_1 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}} \qquad (2)$$

Show that the digital filter coefficients are given by

$$a_0 = a_1 = \frac{2b_p T}{4 + 2b_p T + \omega_p^2 T^2}\ ; \quad b_1 = \frac{2\omega_p^2 T^2 - 8}{4 + 2b_p T + \omega_p^2 T^2}\ ; \quad b_2 = \frac{4 + \omega_p^2 T^2 - 2b_p T}{4 + 2b_p T + \omega_p^2 T^2}$$

b) A digital filter with a centre frequency of 1000 Hz and a bandwidth of 150 Hz is required. Assuming a sampling frequency of 10kHz, compute the digital filter coefficients a_0, a_1, b_1 & b_2 and show that $a_0 = a_1 = 0.04409115$; $b_1 = -1.551$ and $b_2 = 0.918176$

c) Comment on the stability of the digital filter $H(z)$ (see equation 2 above) which you have obtained .

Ans: *Filter is stable.*

Q2)

a) Determine the impulse response $h_d[n]$ of the bandpass filter whose frequency response is given by

$$H_d(\theta) = \begin{cases} e^{-j3\theta} & \frac{\pi}{4} \le |\theta| \le \frac{3\pi}{4} \\ 0 & \text{otherwise} \end{cases}$$

b) To obtain a finite impulse response from $h_d[n]$ a Bartlett window of length $N = 7$ is used. Compute the coefficients of the FIR filter with this impulse response.

Note: The Bartlett Window function is given by

$$w_B[n] = \begin{cases} \frac{2n}{N-1} & 0 \le n < \frac{(N-1)}{2} \\ 2 - \frac{2n}{N-1} & \frac{N-1}{2} \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

where N is the length of the window.

Ans:

$$h_d[n] = \begin{cases} \frac{1}{\pi(n-3)}\left[\sin(n-3)\frac{3\pi}{4} - \sin(n-3)\frac{\pi}{4}\right] & n \ne 3 \\ \frac{1}{2} & n = 3 \end{cases}$$

h[n]	0	-1/(3π)	0	1/2	0	-1/(3π)	0
n	0	1	2	3	4	5	6

Q3) Design a digital filter which meets the following specifications:

Low-pass filter:	0 to 10 kHz passband
Sampling frequency:	$f_s = 100$ kHz
Transition band:	10 kHz to 20 kHz
Stopband attenuation:	-10 dB(Starting at 20 kHz)

The filter must be monotonic in the pass and stop bands, (i.e. no ripple).

$$\text{Ans:} \quad H(z) = \frac{0.0675 + 0.1349z^{-1} + 0.0675z^{-2}}{(1 - 1.143z^{-1} + 0.4123z^{-2})}$$

Q4) Consider the following analogue system with a transfer function

$$H(s) = \frac{\alpha}{s+\alpha} \Rightarrow h(t) = e^{-\alpha t}$$

where α (= 10^4 rad/sec) is the analogue cut-off frequency.

(a) Using bilinear transformation, show that the transfer function *H(z)* is

$$H(z) = \frac{1-a}{2}\left[1 + \frac{(1+a)z^{-1}}{1-az^{-1}}\right] \quad \text{where } a = \frac{2-\alpha T}{2+\alpha T}$$

where $a = 10^4$ rad/sec and the sampling period T is 100 μs.

(i) What is the dc gain of *H(z)*?
(ii) At what frequency is the *H(θ)* equal to zero? (θ - digital frequency).
(iii) Calculate the impulse response *h[n]*.
(iv) Assuming that the impulse response decays to *1/e* of its initial value at $n = N$ samples, show that:

$$N = \frac{\ln\left\{\frac{a}{a+1}\right\} - 1}{\ln(a)}$$

End of Chapter 8