Chapter 6

Chapter 6: Discrete-Time Fourier Transform (DTFT)

The Fourier Transform is a very important tool in many aspects of engineering. In the case of digital signal processing, it allows signals to be represented in terms of their constituent frequency components, i.e. as signals in the ‘frequency domain’. In some cases this means signal processing operations can be executed on frequency domain rather than time domain signals, which can have a number of advantages.

Before proceeding, it is worth noting that in signal processing what is referred to as ‘the Fourier transform’ may in fact be one of four possible types of Fourier transform, illustrated in Table 1. The difference between each type of transform is whether the time domain signal is continuous (i.e. analog) or discrete (i.e. digital), and whether its frequency domain representation is continuous (i.e. a function of real numbers ω or θ) or discrete (i.e. a function of integers n or k)

Table 1. Types of Fourier transforms for signal processing

		Frequency domain signal	
		Continuous	Discrete $X[k]$
Time domain signal	Continuous $x(t)$	Fourier transform $X(\omega)$	Fourier series X_n
	Discrete $x[n]$	Discrete time Fourier transform (DTFT) $X(\theta)$	Discrete Fourier Transform (DFT) $X[k]$

Fourier Transform of an arbitrary discrete signal is also called Discrete Time Fourier Transform (DTFT).

For discrete-time signals, the Fourier transform pair is defined by

$$X(\theta) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\theta}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\theta) e^{jn\theta} d\theta$$

$$x[n] \xleftrightarrow{\text{DTFT}} X(\theta)$$

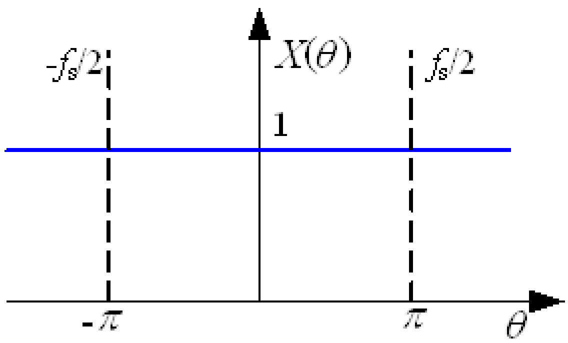
where θ is the digital frequency or relative frequency ($-\pi \leq \theta \leq \pi$) and $\theta = \omega T$. $X(\theta)$ is periodic with period 2π. (ie. $X(\theta + 2\pi) = X(\theta)$). Note that although $x[n]$ is discrete, $X(\theta)$ is a continuous function of frequency θ (hence the integral in the inverse transform).

Note: The Fourier transform of an analogue signal is not periodic.

Example: A unit impulse signal $x[n]$ is transformed into its frequency domain counterpart using the DTFT as follows:

$$x[n] = \delta[n]$$

$$X(\theta) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-jn\theta} = 1$$



Example:

$$x[n] = \begin{cases} 0 & n < 0 \\ a^n & n \geq 0, |a| < 1 \end{cases}$$

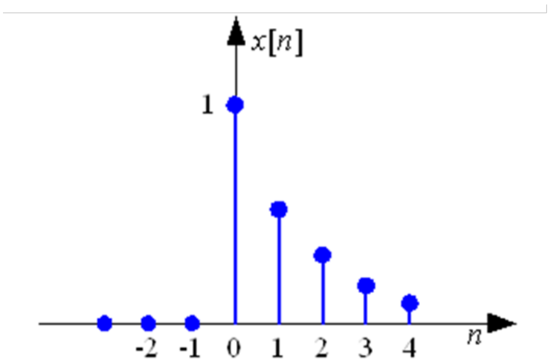
$$X(\theta) = \sum_{n=0}^{\infty} a^n e^{-jn\theta} = \sum_{n=0}^{\infty} (ae^{-j\theta})^n$$

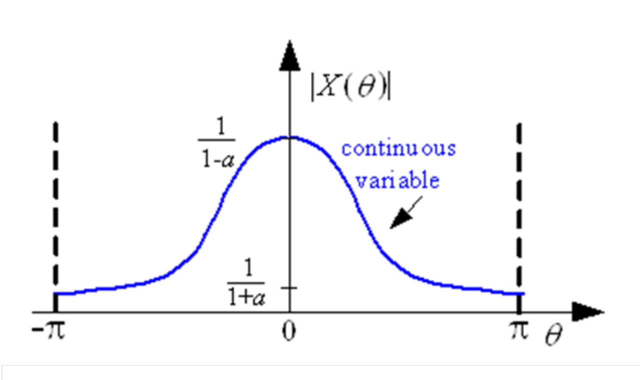
$$= \frac{1}{1 - ae^{-j\theta}}$$

$$X(\theta) = \frac{1}{(1 - a\cos\theta) + ja(\sin\theta)}$$

$$\therefore |X(\theta)| = \frac{1}{\sqrt{1 - 2a\cos\theta + a^2}}$$

$$\angle X(\theta) \overset{\Delta}{=} \phi(\theta) = -\tan^{-1}\left(\frac{a\sin\theta}{1 - a\cos\theta}\right)$$





Note: The above $X(\theta)$ can be obtained using the z-transform as well.

$$x[n] = a^n u[n] \Rightarrow X(z) = \frac{1}{1 - az^{-1}}$$

$$X(\theta) = X(z)\big|_{z=e^{j\theta}} = \frac{1}{1 - ae^{-j\theta}}$$

6.1 Properties of the DTFT

$$x[n] \stackrel{\text{DTFT}}{\rightarrow} X(\theta)$$

$$x_1[n] \stackrel{\text{DTFT}}{\rightarrow} X_1(\theta)$$

$$x_2[n] \stackrel{\text{DTFT}}{\rightarrow} X_2(\theta)$$

Linearity: $ax_1[n] + bx_2[n] \stackrel{\text{DTFT}}{\leftrightarrow} aX_1(\theta) + bX_2(\theta)$

Time Shift: $x[n-k] \stackrel{\text{DTFT}}{\leftrightarrow} e^{-j\theta k} X(\theta)$

Proof: $\left\{ x[n-k] \stackrel{Z}{\leftrightarrow} z^{-k} X(z) \right\}$

$z^{-k} X(z) = e^{-j\theta k} X(\theta)$ as we define $X(e^{j\theta}) \equiv X(\theta)$

Frequency Shift:

$x[n]e^{jn\theta_0} \stackrel{\text{DTFT}}{\leftrightarrow} X(\theta - \theta_0)$ where $X(\theta - \theta_0) \equiv X(e^{j(\theta - \theta_0)})$

Frequency Differentiation: $nx[n] \stackrel{DTFT}{\leftrightarrow} j\dfrac{d}{d\theta} X(\theta)$

Convolution: $x_1[n] * x_2[n] \leftrightarrow X_1(\theta) X_2(\theta)$

Multiplication: $x_1[n] x_2[n] \leftrightarrow X_1(\theta) * X_2(\theta)$

Parseval's Theorem:

$$\sum_{n=-\infty}^{\infty} x_1[n] x_2^*[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\theta) * X_2(\theta) d\theta$$

Exercise:
The frequency response of an ideal high pass filter, in the fundamental interval $-\pi \le \theta \le \pi$ is given by

$$H(\theta) = \begin{cases} 1 & \theta_C \le |\theta| \le \pi \\ 0 & 0 \le |\theta| < \theta_C \end{cases}$$

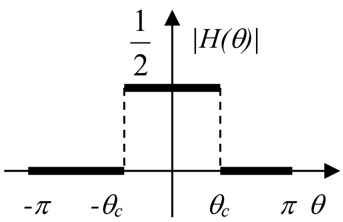
a) Find the impulse response of the ideal high pass filter
b) Sketch the impulse response for $\theta_C = \frac{\pi}{2}$

Example: The frequency response of an ideal low pass filter, in the fundamental interval $-\pi \le \theta \le \pi$ is given by

$$H(\theta) = \begin{cases} \frac{1}{2} & 0 \le |\theta| \le \theta_C \\ 0 & \theta_C < |\theta| \le \pi \end{cases}$$

(a) Find the impulse response of the ideal low pass filter.
(b) Sketch the impulse response for $\theta_C = \frac{\pi}{4}$.

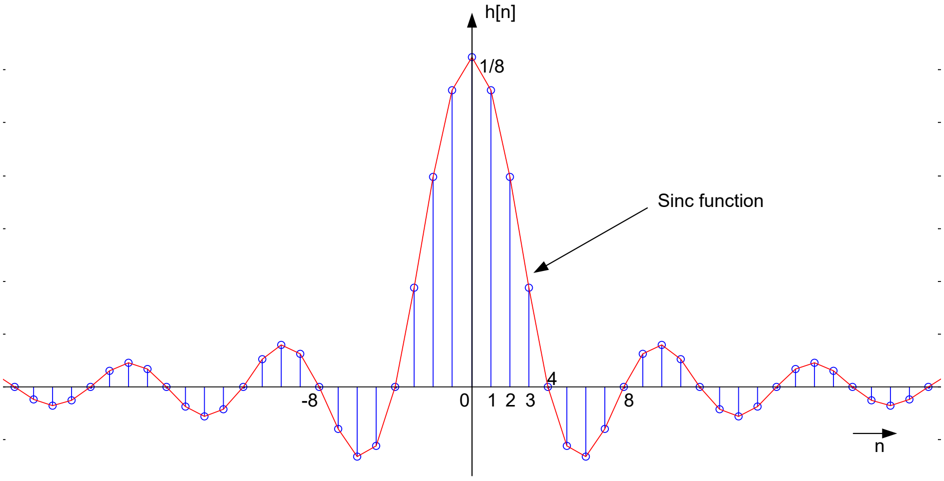
(a)



$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\theta) e^{jn\theta} d\theta = \frac{1}{2\pi}\int_{-\theta_C}^{\theta_C} \frac{1}{2} e^{jn\theta} d\theta = \frac{1}{4\pi}\frac{e^{j\theta_C n} - e^{-j\theta_C n}}{jn}$$

$$= \frac{1}{2\pi}\frac{\sin \theta_c n}{n}$$

(b) $\theta_c = \frac{\pi}{4}$ $\qquad h(n) = \frac{1}{2\pi}\frac{\sin\frac{n\pi}{4}}{n} = \frac{1}{8}\frac{\sin\frac{n\pi}{4}}{\frac{n\pi}{4}}$

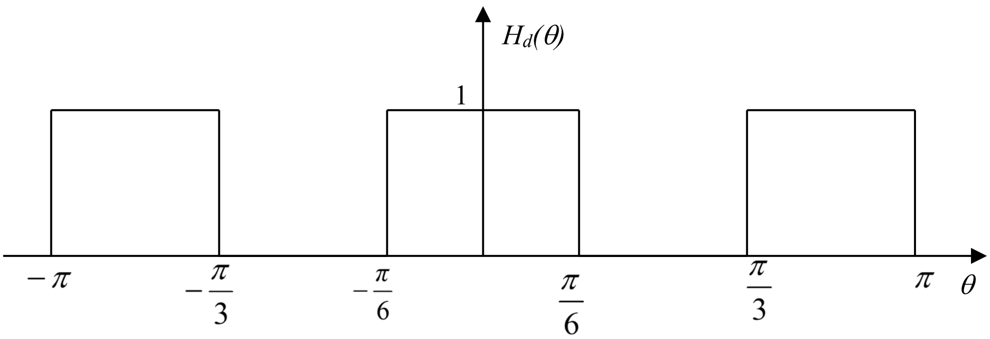


Example: The frequency response of a band stop filter is given by

$$H_d(\theta) = \begin{cases} 1 & |\theta| \le \dfrac{\pi}{6} \\ 0 & \dfrac{\pi}{6} < |\theta| < \dfrac{\pi}{3} \\ 1 & \dfrac{\pi}{3} \le |\theta| \le \pi \end{cases}$$

Show that the impulse response $h_d[n]$ of the band stop filter is given by

$$h_d[n] = \delta[n] - \frac{\sin\left(\frac{\pi}{3}n\right)}{\pi n} + \frac{\sin\left(\frac{\pi}{6}n\right)}{\pi n}$$



$$h_d[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_d(\theta)e^{j\theta n}d\theta=\frac{1}{2\pi}\left\{\int_{-\pi}^{-\frac{\pi}{3}}e^{j\theta n}d\theta+\int_{-\frac{\pi}{6}}^{\pi/6}e^{j\theta n}d\theta+\int_{\frac{\pi}{3}}^{\pi}e^{j\theta n}d\theta\right\}$$

$$=\frac{1}{2\pi}\left\{\left[\frac{e^{j\theta n}}{jn}\right]_{-\pi}^{-\frac{\pi}{3}}+\left[\frac{e^{j\theta n}}{jn}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}+\left[\frac{e^{j\theta n}}{jn}\right]_{\frac{\pi}{3}}^{\pi}\right\}$$

$$=\frac{1}{2\pi}\left\{\frac{e^{-j\frac{\pi}{3}n}-e^{-j\pi n}}{jn}+\frac{e^{j\frac{\pi}{6}n}-e^{-j\frac{\pi}{6}n}}{jn}+\frac{e^{j\pi n}-e^{-j\frac{\pi}{3}n}}{nj}\right\}$$

$$=\frac{\sin\pi n}{\pi n}+\frac{\sin\frac{\pi}{6}n}{\pi n}-\frac{\sin\frac{\pi}{3}n}{\pi n}$$

where $\frac{\sin\pi n}{\pi n}=1 \qquad n=0$ and zero elsewhere

$$h_d[n]=\delta[n]-\frac{\sin\frac{\pi}{3}n}{\pi n}+\frac{\sin\frac{\pi}{6}n}{\pi n}$$

6.2 The Discrete Fourier Transform (DFT)

The importance of the discrete Fourier transform (DFT) for practical applications is hard to overstate. If you've ever seen a real time spectrum of just about anything, chances are it was generated using a DFT. The only practical alternative to the DFT is a bank of analog filters (e.g. as used in a graphic equalizer), however with the availability of cheap processing power, and the fast algorithms for DFT computation, spectrum analysis is a viable component in many present-day systems.

$$x[n]\xleftrightarrow{DTFT}X(\theta) \qquad \text{DTFT}$$

$$x[n]\xleftrightarrow{DFT}X[k] \qquad \text{DFT}$$ {DFT mapping $x[n]$ to another sequence}

The DFT denoted by $X[k]$ (complex valued sequence), is obtained by sampling the Discrete Time Fourier Transform $X(\theta)$ at a finite number of frequency points

This sampling is conventionally performed at equally spaced points over the period extending over $-\pi \leq \theta \leq \pi$.

The DFT allows us to determine the frequency content of a signal, that is, to perform spectral analysis.

The DFT plays a central role in the implementation of a variety of digital signal processing algorithms, as a result of the existence of the efficient algorithm for the Fast Fourier Transform (FFT).

(1) Fourier transform of a discrete signal (DTFT) is

$$X(\theta) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\theta}, \quad -\pi \leq \theta \leq \pi$$

(2) $X(\theta) = X(z)\,|_{z=e^{j\theta}}$

{Evaluate z-transform on the unit circle}

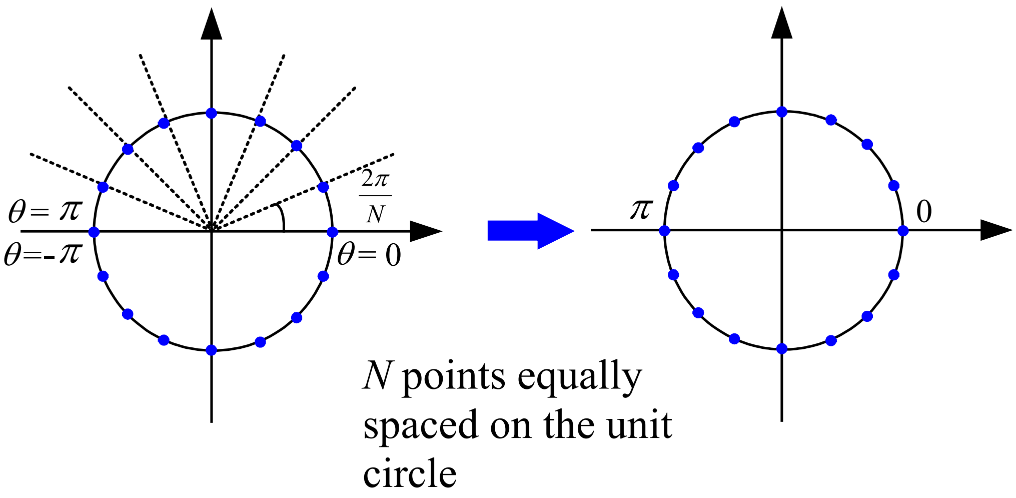
(3) Discrete Fourier Transform (DFT) or N-point DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-jn\left(\frac{2\pi k}{N}\right)}$$

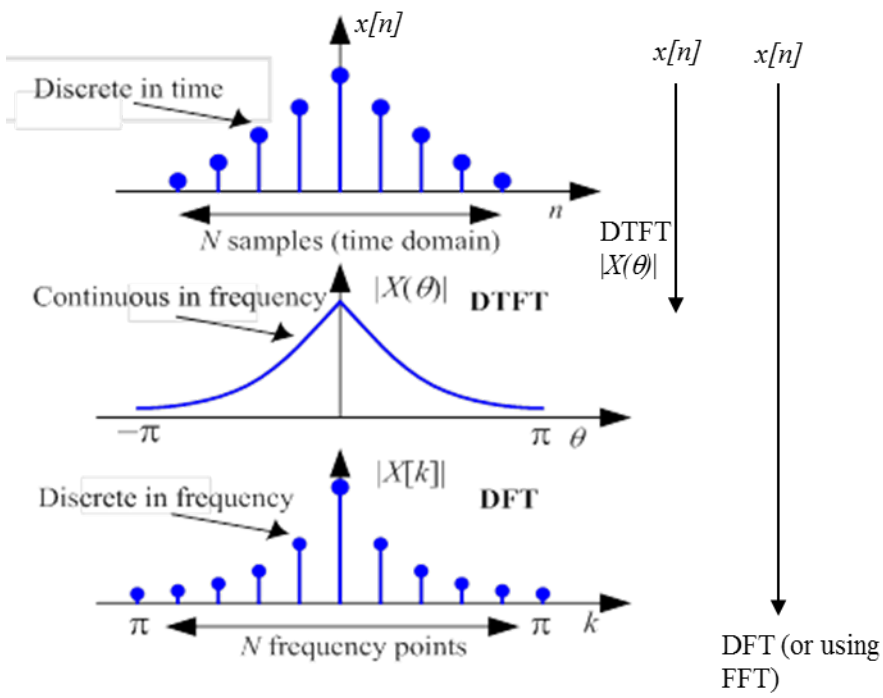
i.e. sampling $X(\theta)$ at N equally spaced interval, $k = 0,1,2,\ldots N-1$.
Note: One of the most important properties of the DFT is implied periodicity.

Number of time samples = N Number of frequency samples (k) = N

The DFT corresponds to sampling the z-transform of $X(z)$ at N-points equally spaced in angle around the unit circle.



Note:



- When the DFT of a block of N samples is calculated, the assumption is that the original signal actually repeats itself periodically, with period N.
- Clearly, for all real signals this will not be true.
- Even for artificial signals such as a pure sinusoid this will only be true if N is a multiple of the period of the sinusoid.
- This "windowing" process introduces a slight distortion into the frequency representation of the signal being analysed.

DFT calculation

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-jn\frac{2\pi k}{N}} \qquad k = 0,1,2,3,....N$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\cos(\frac{2\pi}{N}nk) \; - \; j\sum_{n=0}^{N-1} x[n]\sin(\frac{2\pi}{N}nk)$$

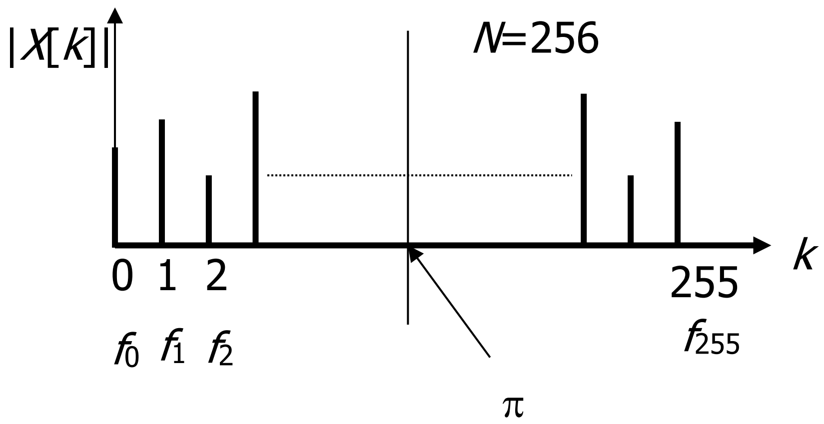
$$X[k] = \quad A[k] \quad - \quad B[k]$$

$$|X[k]| = \sqrt{A[k]^2 + B[k]^2} \qquad k = 0,1,2,3,....N-1$$

$$\Delta\theta = \frac{2\pi}{N}$$

$$\Delta f = \frac{f_s}{N}$$

$$f_k = k(fs/N) = k\,\Delta f \quad (k=0,\ 1,\ 2,\ ....N\text{-}1)$$



The frequency resolution (Δf) can be made as small as desired by increasing the value of N (window size being analysed)

Example: Let f_s = 8000 Hz, number of sample N = 1000

Frequency resolution $\Delta f = \frac{f_s}{N} = \frac{8000}{1000} = 8Hz$

$f_0 = 0$; f_1 = 8Hz, f_2 = 16 Hz, …. f_{999} = 8000Hz

$$X[k] = \sum_{0}^{999} x[n]\, e^{-jn\left(\frac{2\pi k}{1000}\right)}$$

k = 0,1,2,3, …

Example: A speech signal is sampled at a rate of 20000 samples/sec. A sequence of length (*N*) 1024 samples is selected and the 1024-point DFT is computed.

(1) What is the time duration of segment of speech?

Duration = no of samples × sampling period.
= 1024 (1/20000) = 51.2 ms

(2) What is the frequency resolution (spacing in Hz) between the DFT values.

$$\text{Resolution} = \frac{f_s}{N} = \frac{20000}{1024} = 19.531 Hz$$

Example: Consider the finite length sequence.

$$x[n] = \delta[n] + 0.2\delta[n-2]$$

Find the *N*-point DFT of *x*[*n*] for *N* = 50

Solution: $X(z) = 1 + 0.2\, z^{-2} \Rightarrow X(\theta) = 1 + 0.2\, e^{-j2\theta}$

The *N*-point DFT is obtained by evaluating *X(θ)* at *N* points equally spaced around the unit circle.

$$\theta = \frac{2\pi k}{N}$$

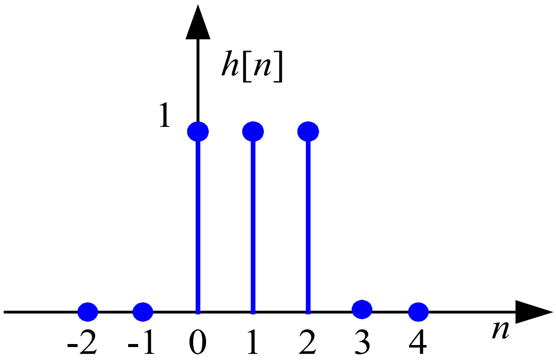
Therefore the DFT is given by:

$$X[k] = 1 + 0.2 e^{-j\left(\frac{2\pi k}{N}\right)2}$$
$$= 1 + 0.2 e^{-j\left(\frac{2\pi k}{25}\right)}$$

k = 0, 1, 2, …. 49

Example: Find the DFT of the three-sample averager.

$$h[n] = \begin{cases} \frac{1}{3} & 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$



$$H[k] = \sum_{n=0}^{N-1} h[n] e^{-jn(\frac{2\pi}{N}k)} = \sum_{n=0}^{2} \frac{1}{3} e^{-jn(\frac{2\pi}{N}k)}$$

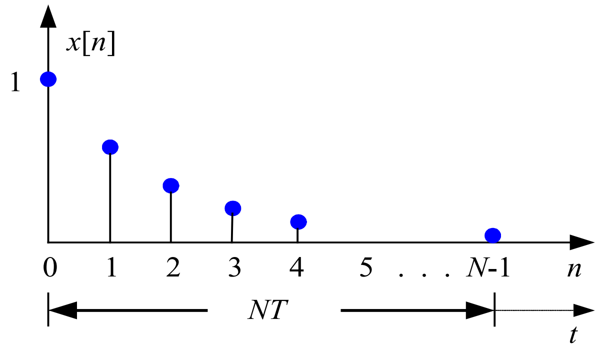
$$H[k] = e^{-j(\frac{2\pi}{N}k)} \frac{(1 + 2\cos(\frac{2\pi k}{N}))}{3}$$

$$|H[k]| = \frac{1}{3}\left|[1 + 2\cos(\frac{2\pi k}{N})]\right| \qquad \text{where } k = 0,1,2$$

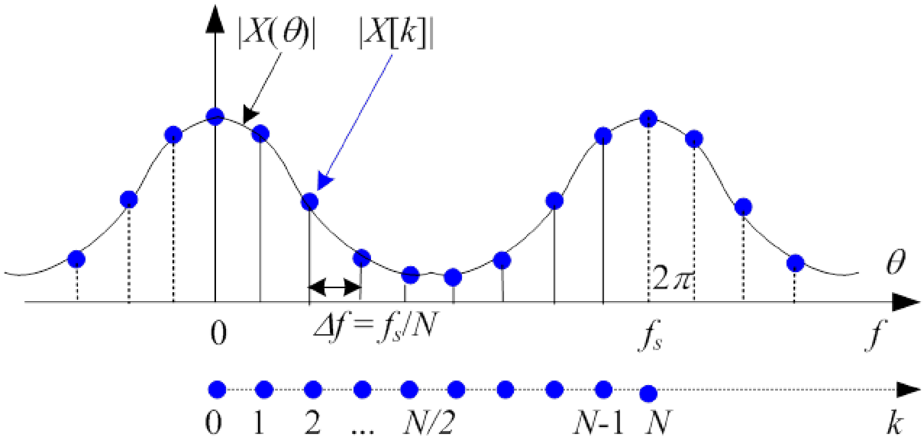
Exercise: Compute the N-point DFT $H[k]$ of the sequence $h[n]$. Show that when $N = 8$ the value of $H(2) = -1$.

$$h[n] = \begin{cases} 1 & -3 \le n \le 3 \\ 0 & \text{otherwise} \end{cases}$$

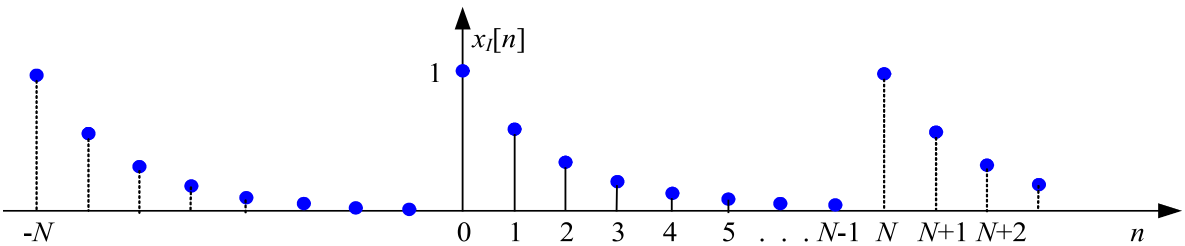
Example: Find the DTFT and DFT of a truncated exponential sequence $x_a(nT) = a^{nT}$, $n = 0,1,\ldots, N-1$, $|a| < 1$



This has DTFT $X(e^{j\theta}) = \dfrac{1 - a^{NT} e^{-j\theta N}}{1 - a^{T} e^{-j\theta}}$



and DFT $X[k] = \dfrac{1 - a^{NT}}{1 - a^{T} e^{-j\frac{2\pi k}{N}}}$. Note also that the inverse DFT $x_I[n]$ (below) is identical to $x[n]$ for n = 0, 1, …, N-1, but outside this region creates periodic repetitions of the truncated $x[n] = x_a(nT)$.



6.2.1 Conjugate Symmetry

The DFT of a real sequence possesses conjugate symmetry about the origin with

$$X[-k] = X^*[k]$$

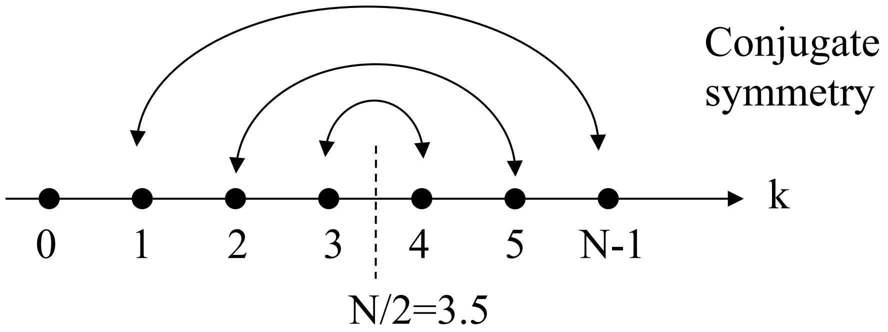
Since the DFT is periodic,

$$X[-k] = X[N-k]$$

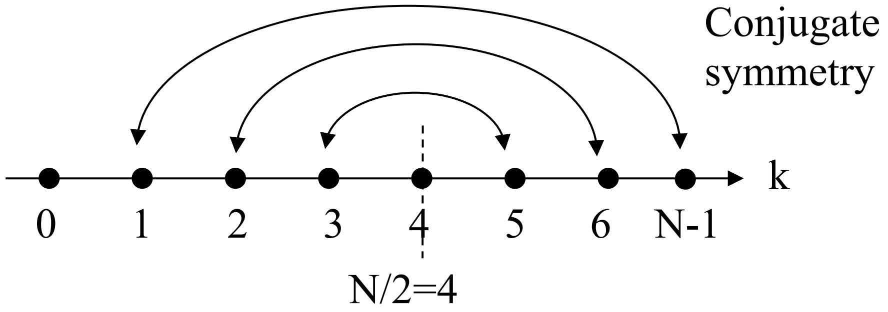
This implies conjugate symmetry about the index $k = \frac{N}{2}$

$$\therefore X[-k] = X^*[k] = X[k-K]$$

<u>N is odd</u> (N=7)



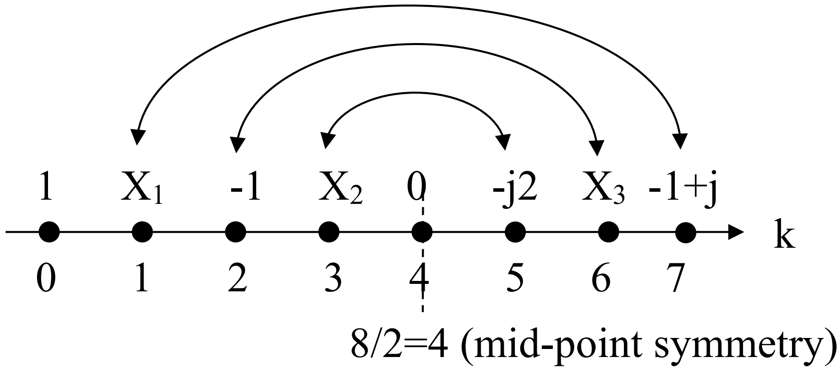
<u>N is even</u> (N=8)



Example: The DFT of a real signal is

{1, X_1, -1, X_2, 0, -j2, X_3, -1+j}

Find X_1, X_2 and X_3.



$$X_1 = x^*[1] = -1 - j$$

$$X_2 = x^*[5] = j2$$

$$X_3 = x^*[2] = -1$$

Note: 8 frequency points = 8 time samples (when IDFT is applied)

6.2.2 Parseval's Theorem

DFT is an energy-conserving transformation that allows to find the signal energy from the signal or its spectrum. Therefore, the sum of squares of the signal samples is related to the sum of squares of the magnitude of the DFT samples.

Example:

The DFT of a real signal is $\{1, X_1, -1, X_2, -7, -j2, X_3, -1+j\}$. What is the signal energy?

($k=0$ marks the first element, 1; $k=7$ marks the last element, $-1+j$.)

X_1, X_2 and X_3 can be found first using conjugate symmetry property

$$\text{Signal Energy} = \frac{1}{8}\sum |X[k]|^2$$

$$= \frac{1}{8}\left[1^2 + (\sqrt{2})^2 + (-1)^2 + 2^2 + 7^2 + 2^2 + 1^2 + \left(\sqrt{2}\right)^2\right]$$

$$= \frac{1}{8}[1+2+1+4+49+4+1+2]$$

$$= 8$$

Exercise:

a) Determine the 4-point DFT of a signal x[n]=[1 -1 -1 1].
b) Find the energy of x[n] using the information in time domain.
c) Find the energy of x[n] using its spectrum. Show that it is same as answer in question (b).

6.3 Inverse Discrete Fourier Transform (IDFT)

The inverse DFT equation comes directly from the Inverse Fourier Transform equation

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{jn\frac{2\pi}{N}k} \quad 0 \le n \le N\text{-}1$$

However, the signal which it produces will be a periodic signal repeating every N samples

Example:

If $X[k]$ = {4, -j2, 0, j2}, find its IDFT
N=4

$$\therefore x[n] = \frac{1}{4}\sum_{k=0}^{3} X[k]e^{j\frac{2\pi kn}{4}}$$

$$= \frac{1}{4}\sum_{k=0}^{3} X[k]e^{j\frac{\pi kn}{2}}$$

$$n = 0 \quad x[0] = \frac{1}{4}[4 - j2 + 0 + j2] = 1$$

$$n = 1 \quad x[1] = \frac{1}{4}[4 - j2e^{j\frac{\pi}{2}} + 0 + j2e^{j\frac{3\pi}{2}}] = 2$$

$$n = 2 \quad x[2] = \frac{1}{4}[4 - j2e^{j\pi} + 0 + j2e^{j3\pi}] = 1$$

$$n = 3 \quad x[3] = \frac{1}{4}[4 - j2e^{j\frac{3\pi}{2}} + 0 + j2e^{j\frac{9\pi}{2}}] = 0$$

The IDFT is $x[n]$={1, 2, 1, 0}
↑
n=0

Use of DFT

- Clearly, the DFT is useful in that it allows the spectral content of a signal to be determined
- Additionally, once in the frequency domain the DFT of a signal can be processed in order to "filter" or "alter" the signal is some desired fashion
- The IDFT can then be used to regenerate the processed signal

Note: If any window, other than rectangular, has been used then DFT and IDFT blocks must overlap by 50%, if perfect reconstruction is required

6.4 Padding with Zeros and Frequency Resolution

$$DFT: \quad X[k] = \sum_{n=0}^{N-1} x[n] e^{-jn\frac{2\pi k}{N}} \qquad k = 0,1,2,3,....N$$

- The frequency resolution depends directly on the number of time samples N, i.e. the longer the sequence $x[n]$, the more frequency points in the DFT domain (frequency domain).

- To obtain more points in the DFT sequence, we can always increase the duration of $x[n]$ by adding additional zero-valued elements. This procedure is called padding with zeros.

- These zero-valued elements contribute nothing to the sum in the above equation, but act to decrease the frequency spacing ($2\pi/N$).

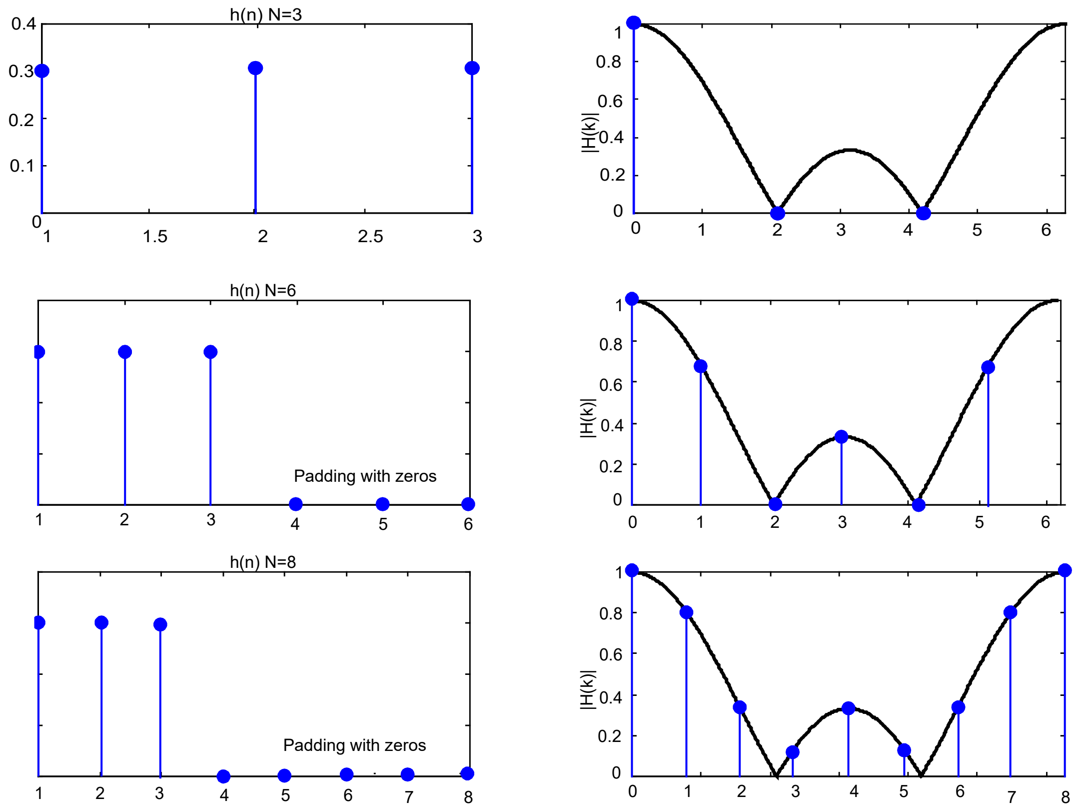
- The zero padding gives us a high-density spectrum and provided a better displayed version for plotting. But it does not give us a high resolution spectrum because no new information is added to the signal. Only additional zeros are added in the data.

Example: The three sample averager can be transformed using both the DTFT and the DFT. Because the DTFT is determined for all $n \in [-\infty,\infty]$, it is unaffected by the window length N.

On the other hand, adding zeros to the three sample averager gives a different view of the magnitude spectrum, depending on the length N.

Lengths of $N = 3$, 6 and 8 are shown below. Note that although the frequencies at which the DFT is evaluated are different in each case, the underlying spectrum (given by the DTFT) is exactly the same.

Three Sample Averager



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Fast Fourier Transform

- The Fast Fourier Transform (FFT) is simply a mathematical technique to accelerate the calculation of the DFT. It was developed by Cooley and Tukey (1965) requires N to be a power of 2.

- Typically, if the DFT is calculated for a block of $2n$ samples e.g. 512 or 1024 samples (N) it would make the calculation of the DFT quite demanding.

- The FFT simply uses repetition and redundancy in the calculation to speed it up.

- The FFT is simply a TECHNIQUE to calculate the DFT, NOT a different transform.

CHAPTER 6: PROBLEM SHEET 6

Q1) A speech signal is sampled with a sampling period of 125ms. A frame of 256 samples is selected and a 256 point DFT is computed. What is the spacing between the DFT values in Hz.

Ans: 0.03125Hz

Q2) Compute the N-point DFT, *H[k]* of the sequence *h[n]*. Show that when $N = 3$ the value of $H(3) = 3/5$

$$h[n] = \begin{cases} \frac{1}{5} & -1 \le n \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Ans: $H[k] = \frac{1}{5}\left(2\cos\left(\frac{2\pi k}{3}\right) + 1\right)$

Q3) A 5 kHz sinusoidal signal is sampled at 40 kHz and 128 samples are collected and used to compute the 128-point DFT of the signal. What is the time duration in seconds of the collected samples? At what DFT indices do we expect to see any peaks in the spectrum?

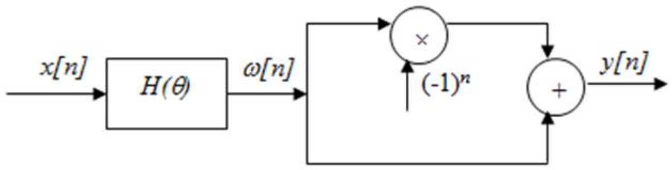
Ans: 16 and 112

Q4) Consider the finite length sequence: $x[n] = \delta[n] + 0.2\delta[n-2]$
Write the equation for the N-point DFT of $x[n]$ for $N = 25$

Ans: $X[k] = 1 + 0.2e^{\frac{j2\pi k}{25}}$

Q5) For the system in the figure below, sketch the output $y[n]$ when the input $x[n]$ is $\delta[n]$ and $H(\theta)$ is an ideal lowpass filter as follows:

$$H(\theta) = \begin{cases} 1 & 0 \le |\theta| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\theta| \le \pi \end{cases}$$



Ans: $\omega(n) = \frac{1}{\pi}\left(\frac{\sin\frac{n\pi}{2}}{n}\right)$; $y(n) = \frac{1}{2}\left[(-1)^n + 1\right]\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}}$

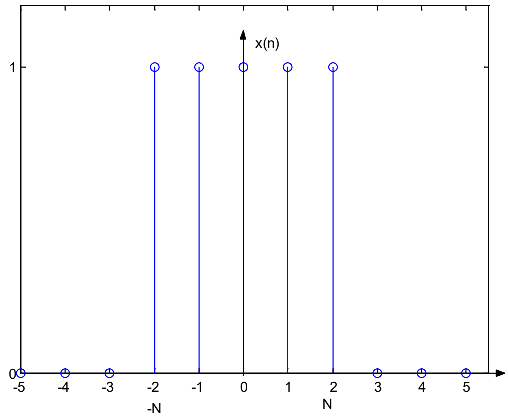
Q6) Let x[n]={1, 2, 1, 0}. With N=4, compute the 4-point DFT.
↑ n=0

Ans: $X[k]=\{4, -j2, 0, j2\}$

Q7) Find the 8-point DFT of $x[n]$ using conjugate symmetry property of $X[k]$. x[n]={1, 1, 0, 0, 0, 0, 0, 0}.

Ans: $X[k]=\{2, 1.707-j0.707, 1-j, 0.293-j0.707, 0, 0.293+j0.707, 1+j, 1.707+j0.707\}$

Q8)



If $x[n] = \begin{cases} 1 & |n| \le N \\ 0 & |n| > N \end{cases}$

Determine $X(\theta)$.

$$ans : X(\theta) = \frac{\sin\left\{\frac{(2N+1)\theta}{2}\right\}}{\sin\left(\frac{\theta}{2}\right)}$$

End of Chapter 6