Chapter 3

Chapter 3: Discrete-Time Systems

3.1 Introduction

A discrete-time system is a device or algorithm that operates on a discrete-time signal called the input or excitation according to some well defined rule, to produce another discrete-time signal called the output or response.

We say that the input signal $x[n]$ is transformed by the system into a signal $y[n]$, and express the general relationship between $x[n]$ and $y[n]$ as

$$\boxed{y[n] = H\{x[n]\} \qquad (3.1)}$$

where the symbol H denotes the transformation or processing performed by the system on $x[n]$ to produce $y[n]$ (see Figure 3.1).

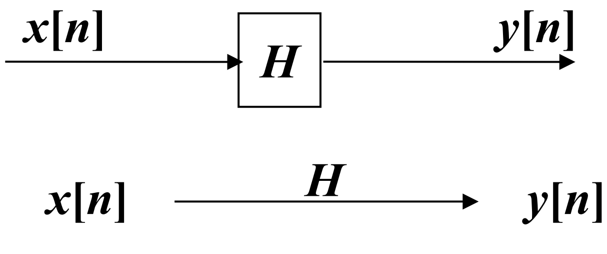


Figure 3.1: Block diagram representation of a discrete-time system

3.2 Block Diagram Representation

In order to introduce a block diagram representation of discrete-time systems, we need to define some basic blocks that can be interconnected to form complex systems.

3.2.1 An adder

A system that performs the addition of two signal sequences to form another sequence, which we denote as $y[n]$.

Note: It is not necessary to store either one of the sequences in order to perform the addition. In other words, the addition operation is **memoryless**.

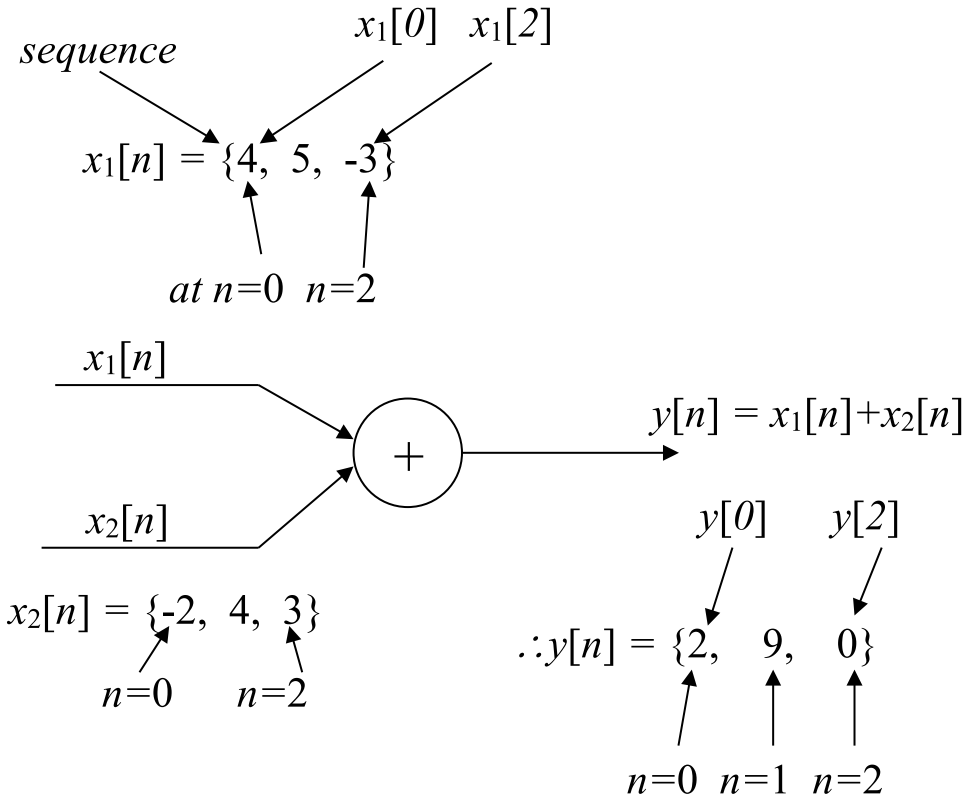


Figure 3.2: Block diagram representation of an adder, $x_1[n]$ and $x_2[n]$ denote discrete-time input signals and $y[n]$ denote a discrete-time output signals.

3.2.2 A constant multiplier

This operation simply represents applying a scale factor on the input *x[n]*. Note that this operation is also **memoryless**.

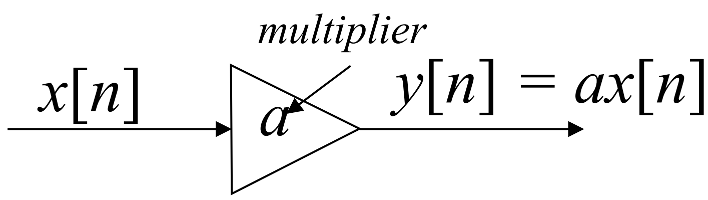


Figure 3.3: Block diagram representation of a multiplier. x[n] and y[n] denote discrete-time input and output signals respectively. 'a' denotes a scalar multiplier.

Example:

sequence

$$x[n] = \{2, -5, 6, 8\};\ a = 0.1;\ y[n] = \{0.2, -0.5, 0.6, 0.8\}$$

n=0 *n=3* *n=0* *n=3*

3.2.3 A Unit Delay Element

The unit delay is a special system that simply delays the signal passing through it by one sample.

If the input signal is $x[n]$, the output is $x[n-1]$. In fact, the sample $x[n-1]$ is stored in memory at time $n-1$ and it is recalled from memory at time n to form $y[n] = x[n-1]$.

Thus this basic building block requires memory. We use the symbol T or z^{-1} to denote the unit of delay.

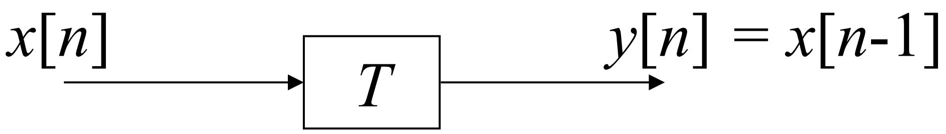
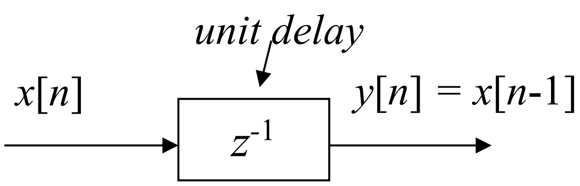


Figure 3.4: Block diagram representation of a unit delay.T denotes the sampling period.

Example:



$n = -1;\ n=0;\ n=1$

$$x[n] = \{0, 1, 0, 5, 7, -2, -1, 0\}$$

$$y[n] = \{0, 1, 0, 5, 7, -2, -1\}$$

$y[0]$ $y[1]$ $y[2]$ ... $y[6]$

{ $y[n] = x[n-1]$; $y[0]=x[0-1]=x[-1]$; $y[1] = x[1-1]=x[0]$;…}

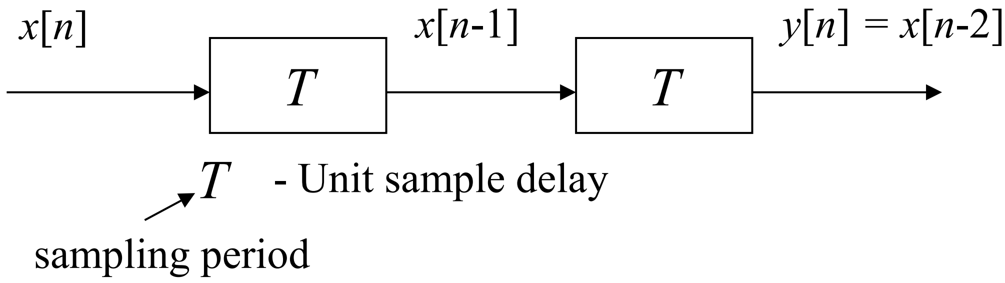
Note: Normally a combination of adders, multipliers and unit delays form a complex discrete-time system.

3.3 Difference Equations

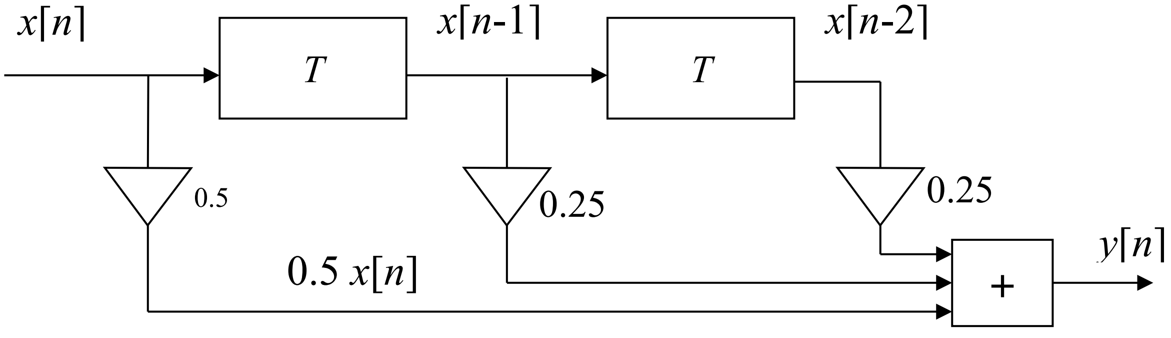
A discrete-time system consisting of combinations of adders, multipliers and unit delays can always be described by a set of difference equations. The equations would be ordinary algebraic equations if no delays were present.

Examples:

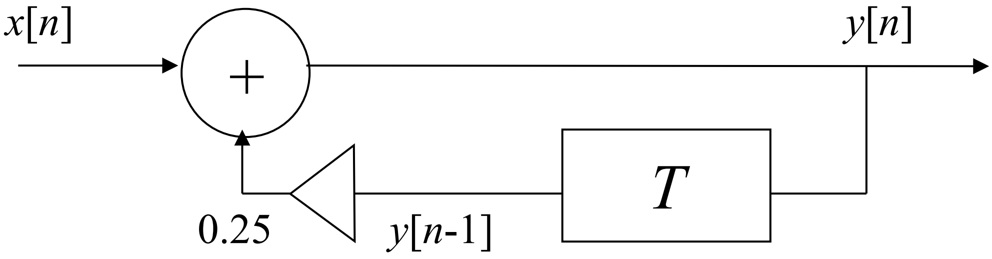
(a) $y[n] = x[n-2]$



(b) $y[n] = \frac{1}{2}x[n] + \frac{1}{4}x[n-1] + \frac{1}{4}x[n-2]$



(c) $y[n] = x[n] + 0.25y[n-1]$



Exercise:

Draw a system implementation for the difference equation as given below:

$$y[n] = x[n] + x[n-1] + x[n-2] + x[n-3]$$

Example $y[n] = a_0x[n] + a_1x[n-1] - b_1y[n-1]$

Draw a system implementation for the above difference equation.

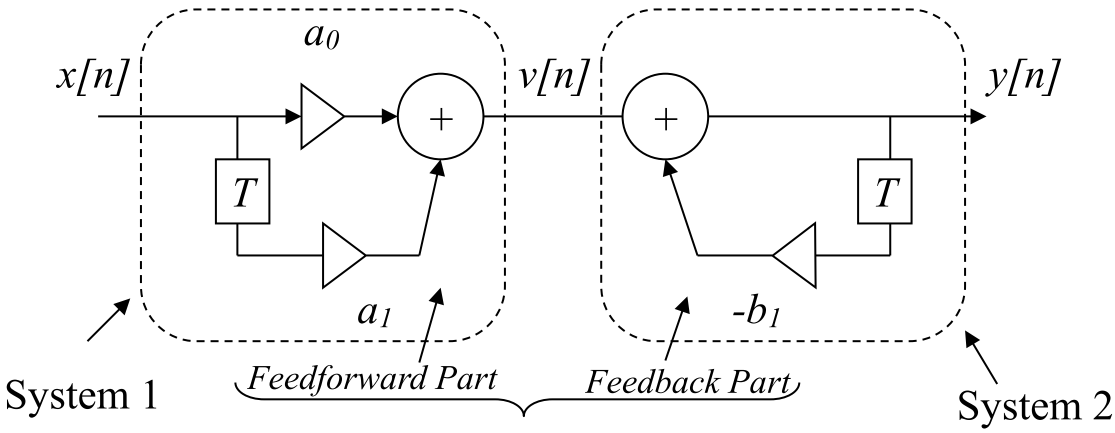


Figure 3.5: Direct Form I structure

We can write the above difference equation as a set of two equations

$$v[n] = a_0x[n] + a_1x[n-1] \quad \text{- system 1}$$
$$y[n] = v[n] - b_1y[n-1] \quad \text{- system 2}$$

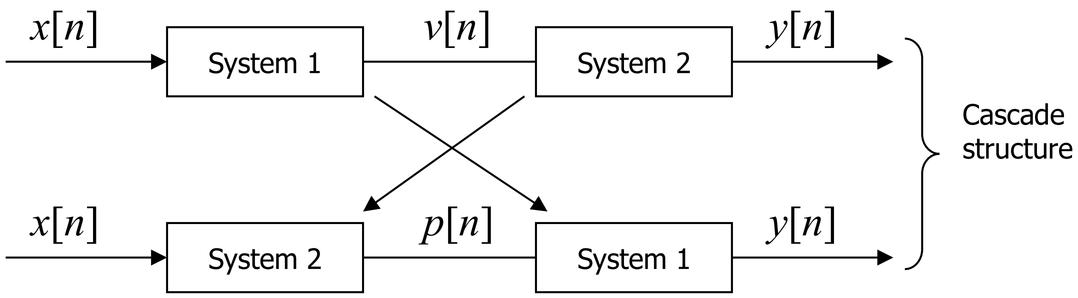


Figure 3.6: Two systems forming a cascade structure can be interchanged without affecting the final output signal.

Without changing the input-output relationship, we can reverse the ordering of the two systems in the cascade representation.

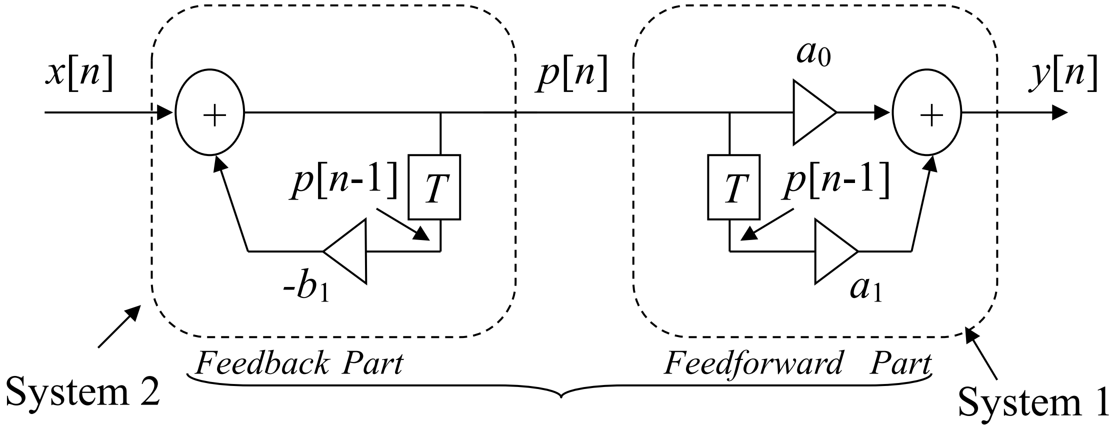


Figure 3.8: Direct Form II structure

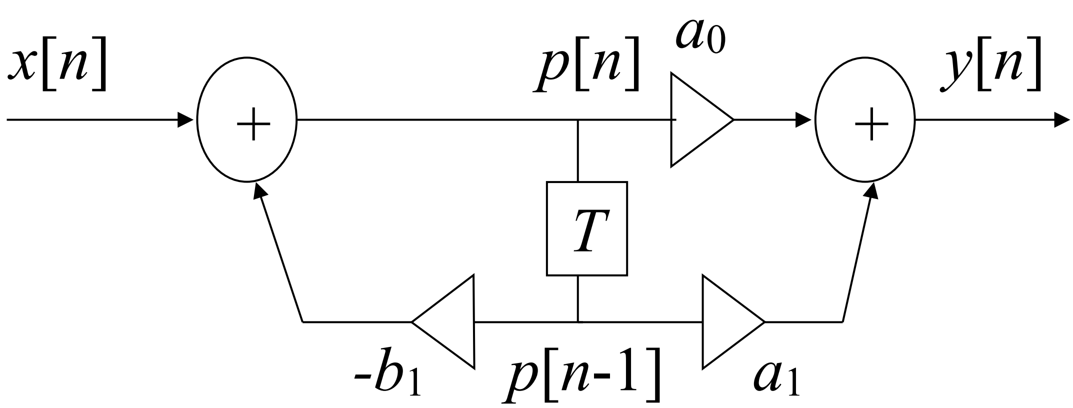


Figure 3.8: Canonic form.

There is no need for two delay operations; they can be combined into a single delay as shown in Figure 3.8. Since delay operations are implemented with memory in a computer, the implementation of Figure 3.8 would require less memory compared to the implementation of Figure 3.8.

It can be proven that both block diagrams Figure 3.5 and Figure 3.8/Figure 3.8 represent the same difference equation.

Proof:

$$y[n] = a_0 x[n] + a_1 x[n-1] - b_1 y[n-1] \tag{3.2}$$

From Figure 3.8:

$$p[n] = x[n] - b_1 p[n-1] \tag{3.2.a}$$

$$y[n] = a_0 p[n] + a_1 p[n-1] \tag{3.2.b}$$

Substituting $n \rightarrow n$-1 in equation (3.2.b),

$$y[n-1] = a_0 p[n-1] + a_1 p[n-2] \tag{3.2.c}$$

Multiplying equation (3.2.c) by b_1,

$$b_1 y[n-1] = a_0 b_1 p[n-1] + a_1 b_1 p[n-2] \tag{3.2.d}$$

Adding equation (3.2.b) and (3.2.d),

$$y[n] + b_1 y[n-1] = \underbrace{a_0 p[n] + a_0 b_1 p[n-1]}_{a_0 x[n]} + \underbrace{a_1 p[n-1] + a_1 b_1 p[n-2]}_{a_1 x[n-1]}$$

Therefore,

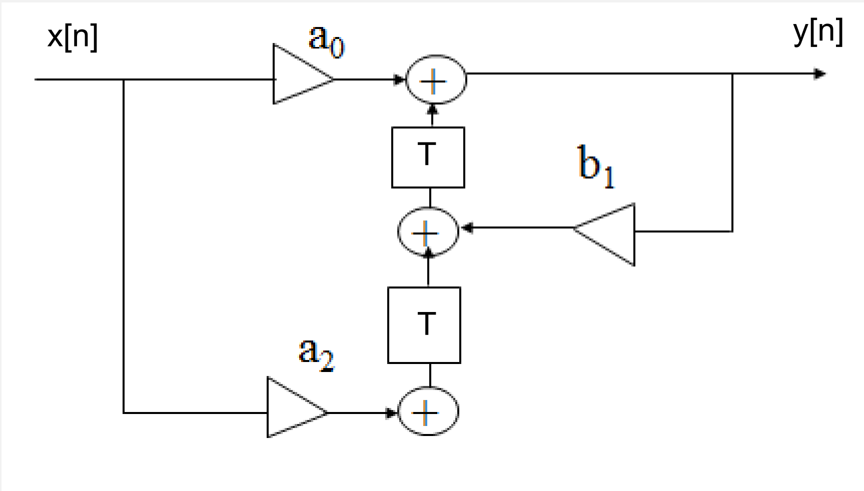
$$y[n] = a_0 x[n] + a_1 x[n-1] - b_1 y[n-1]$$

as in equation (3.2)

Exercise:

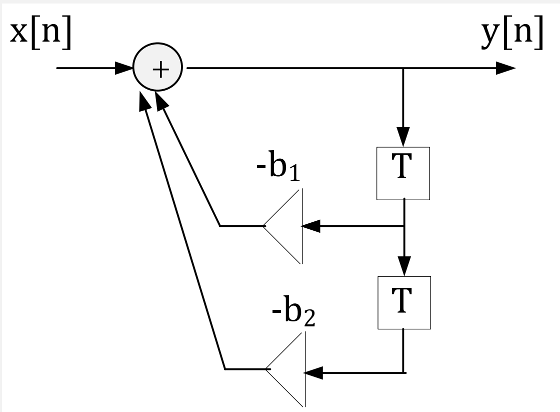
For the given systems shown below, write the difference equation.

a)



Ans: $y[n]=a_0x[n]+ a_2x[n-2]+ b_1y[n-1]$

b)



Ans: $y[n]=x[n]- b_1y[n-1]- b_2y[n-2]$

c) Draw a canonic structure for the difference equation given below:

$$y[n] = x[n] - a_2x[n-2] + b_1y[n-1] - b_2y[n-2]$$

3.4 Classification of Discrete-Time Systems

In the analysis as well as in the design of systems, it is desirable to classify the systems according to the general properties that they satisfy. For a system to possess a given property, the property must hold for every possible input signal to the system. If a property holds for some input signals but for others, the system does not possess the property.

General Categories are:

- Static systems
- Time - invariant systems
- Linear systems
- Causal systems
- Stable systems

3.4.1 Static systems

A discrete-time system is called static or memoryless if its output at any instant '*n*' depends at most on the input sample at the same time, but not on past or future samples of the input.

Example:

$$y[n] = ax[n]$$

$$y[n] = nx[n] + bx^3[n]$$

Both are static or memoryless.

On the other hand, the systems described by the following input-output relations, such as

$$y[n] = x[n] + 3x[n-1]$$

$$y[n] = \sum_{k=0}^{N} x[n-k]$$

are dynamic systems or system with memory.

3.4.2 Time-invariant systems

A time-invariant system is defined as follows:

$$x[n-n_0] \xrightarrow{H} y[n-n_0]$$

where $y[n] = H\{x[n]\}$.

Specifically, a system is time invariant if a time shift in the input signal results in an identical time shift in the output signal.

Example: Determine if the system is time variant or time invariant.

$$y[n] = H\{x[n]\} = nx[n] \qquad (3.3)$$

The response of this system to $x[n\text{-}k]$ is

$$w[n] = nx[n-k]$$

Now if we delay $y[n]$ in (3.3) by k units in time, we obtain

$$\begin{aligned} y[n-k] &= (n-k)x[n-k] \\ &= nx[n-k] - kx[n-k] \end{aligned}$$

This system is time variant, since

$$y[n-k] \neq w[n]$$

Exercise:

Examine the time-invariant properties of the following systems:

a) y[n]= nx(n)
b) y[n]=|x[n] |

Ans: a) Time variant b) Time invariant

3.4.3 Linear Systems

A linear system is defined as follows:

$$\boxed{a_1x_1[n]+a_2x_2[n]\xrightarrow{H}a_1y_1[n]+a_2y_2[n] \qquad (3.4)}$$

where a_1 and a_2 are arbitrary constants.

Example: Three sample averager

$$y[n]=\frac{1}{3}\{x[n+1]+x[n]+x[n-1]\}=H\{x[n]\}$$

$$x[n]\xrightarrow{H}y[n]$$

$$\begin{aligned}H\{ax_1[n]+bx_2[n]\} &= \frac{1}{3}\{ax_1[n+1]+bx_2[n+1]\\ &\quad +ax_1[n]+bx_2[n]\\ &\quad +ax_1[n-1]+bx_2[n-1]\}\\ &=[ay_1[n]+by_2[n]]\end{aligned}$$

The 3-sample averager is a linear system.

Example:

$$y[n]=H\{x[n]\}=x^2[n]$$

$$\begin{aligned}H\{ax_1[n]+bx_2[n]\}&=\{ax_1[n]+bx_2[n]\}^2\\ &=a^2x_1^2[n]+b^2x_2^2[n]+2abx_1[n]x_2[n]\end{aligned}$$

which is not equal to $ax_1^2[n]+bx_2^2[n]$. This system is nonlinear.

Example:

$$\begin{aligned}y[n]&=nx[n]=H\{\ x[n]\ \}\\ H\{\ ax_1[n]+bx_2[n]\ \}&=anx_1[n]+bnx_2[n]\\ &=ay_1[n]+by_2[n]\end{aligned}$$

The system is linear.

3.4.4 Causal systems

A system is said to be <u>causal</u> if the output of the system at any time 'n' depends only on present and past inputs, but does not depend on future inputs. If a system does not satisfy this definition, it is called <u>*noncausal.*</u> Such a system has an output that depends not only on present and past inputs but also on future inputs.

<u>Example:</u>

$$y[n] = x[n] - x[n-1] \quad \rightarrow \text{Causal}$$
$$y[n] = ax[n] \quad \rightarrow \text{Causal}$$
$$y[n] = x[n] + 3x[n+4] \rightarrow \text{Noncausal}$$
$$y[n] = x[-n] \quad \rightarrow \text{Noncausal}$$

{Let n = -1 $\Rightarrow$ y[-1]= x [1], the output at n = -1 depends on the input at n = 1.}

Discrete - time sequence is called **causal** if it has zero values for n<0.

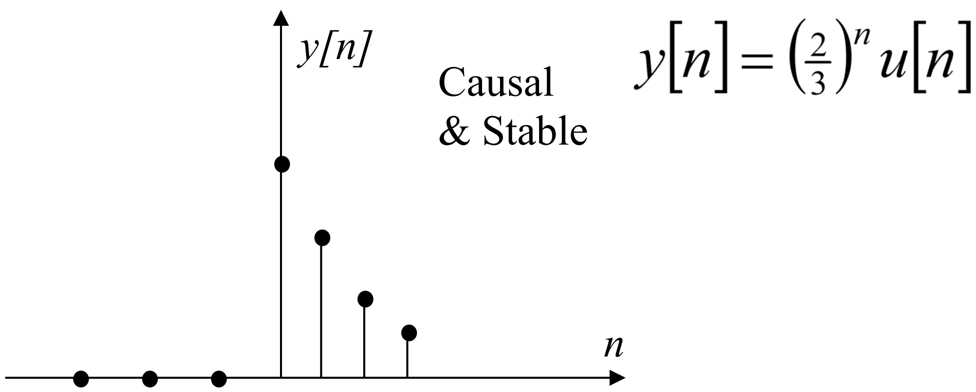


Figure 3.9: An example of causal discrete-time sequence.

3.4.5 Stable Systems

- A discrete signal $x[n]$ is bounded if there exists a finite M such that $|x[n]| < M$ for all n.

- A discrete-time system in Bounded Input-Bounded Output (**BIBO**) stable if every bounded input sequence $x[n]$ produced a bounded output sequence.

$$\text{If } x[n]_{\max} \le A, \text{ then } y[n]_{\max} \le B$$

Example:

The discrete-time system

$$y[n] = ny[n-1] + x[n], \quad n > 0$$

is at rest [i.e. $y[-1]=0$]. Check if the system is BIBO stable.

If $x[n]=u[n]$, then $|x[n]| \le 1$. But for this bounded input, the output is

$$\begin{aligned} n = 0 &\rightarrow y[0] = x[0] = 1 \\ n = 1 &\rightarrow y[1] = 1y[0] + x[1] = 2 \\ n = 2 &\rightarrow y[2] = 2y[1] + x[2] = 5 \\ \vdots & \qquad \vdots \end{aligned}$$

which is unbounded. Hence the system is unstable.

y[0]=1 → y[1]=2 → y[2]=5 → increasing

Exercise: A discrete-time system can be (1) Static or Dynamic, (2) Linear or nonlinear, (3) Time invariant or time varying, (4) Causal or noncausal, (5) Stable or unstable

Examine the following system with respect to the properties above.

(a) $y[n] = e^{ax[n]}$

(b) $y[n] = x[n] + nx[n+1]$

Ans: a) Static, nonlinear, time invariant, casual, stable
b) Dynamic, linear, time varying, noncasual, unstable

3.5 Linear Time-Invariant Discrete (LTD) Systems

3.5.1 Transformation of Discrete-Time signals

A discrete-time signal, $x[n]$ may be shifted in time (delayed or advanced) by replacing the variables n with n-k where $k > 0$ is an integer

x[n-k] => $x[n]$ is delayed by k samples
$x[n+k]$ => $x[n]$ is advanced by k samples

For example consider a shifted version of the unit impulse function (see Figure 3.10). If we multiply an arbitrary signal $x[n]$ by this function, we obtain a signal that is zero everywhere, except at $n = k$.

$$\therefore\ y[n] = x[n]\cdot\delta[n-k] = x[k]\cdot\delta[n-k] \qquad (3.5)$$

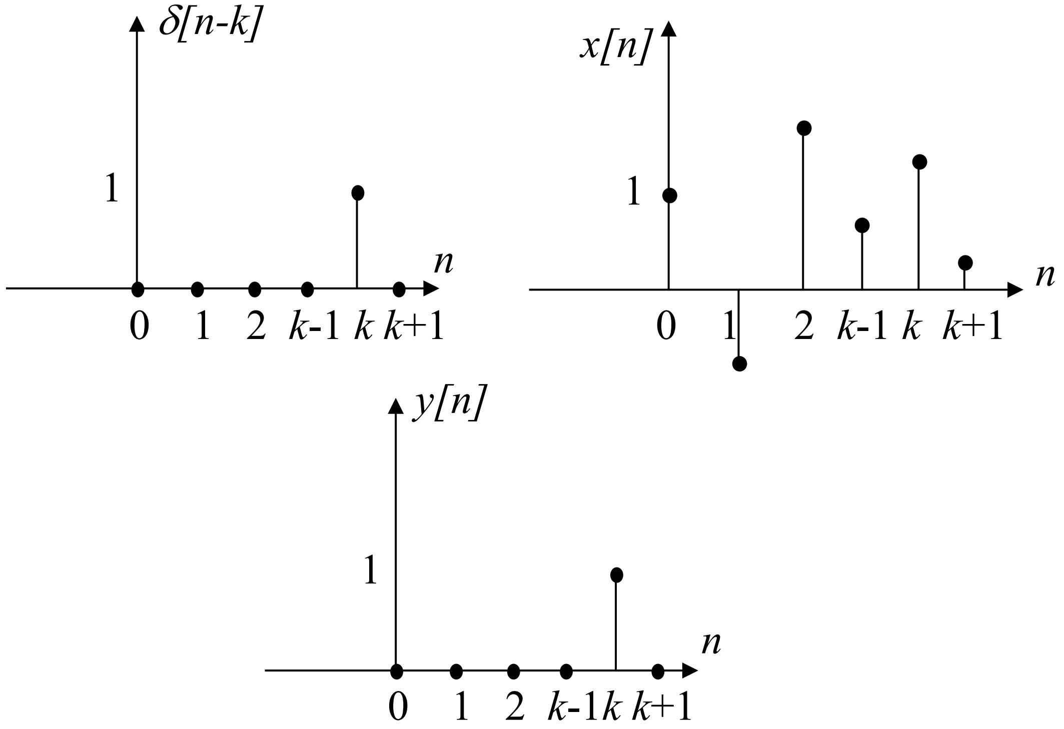
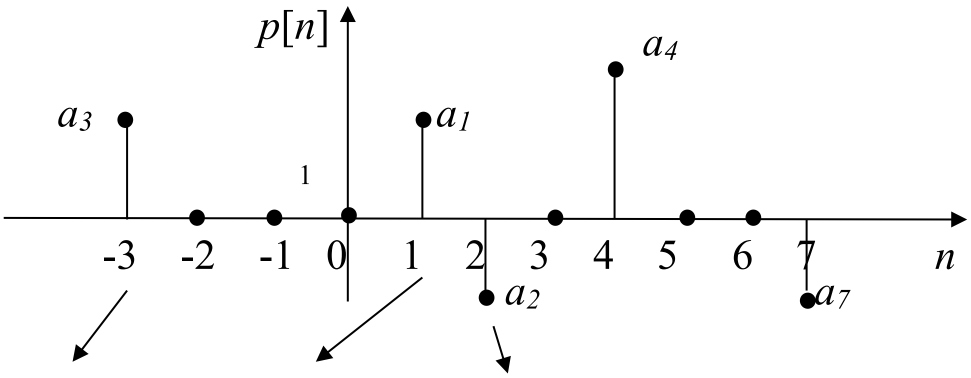


Figure 3.10: Multiplying a discrete-time signal, x[n], with a shifted unit impulse function, δ[n-k], produces a discrete-time signal whose sample is zero except at n=k.

An arbitrary sequence can then be expressed as a sum of scaled and delayed unit impulses.



$p[n] = a_{-3}\delta[n+3] + a_1\delta[n-1] + a_2\delta[n-2] + a_4\delta[n-4] + a_7\delta[n-7]$

Figure 3.11: An example of expressing arbitrary discrete-time sequences as a sum of scaled and delayed unit impulses.

More generally, the discrete-time sequence can be expressed according to

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \qquad (3.6)$$

For real time signals

$$x[n] = \sum_{k=0}^{\infty} x[k]\delta[n-k]$$

and for a real-time signal with a finite number of samples N.

$$x[n] = \sum_{k=0}^{N-1} x[k]\delta[n-k] \qquad (3.7)$$

If $x[n]$ has finite duration, the infinite sum in equation (3.7) may be replaced by a finite sum. That is if $x[n] \neq 0$ for $-N_2 \leq n \leq N_1$.

$$x[n] = \sum_{k=-N_2}^{N_1} x[k]\delta[n-k]$$

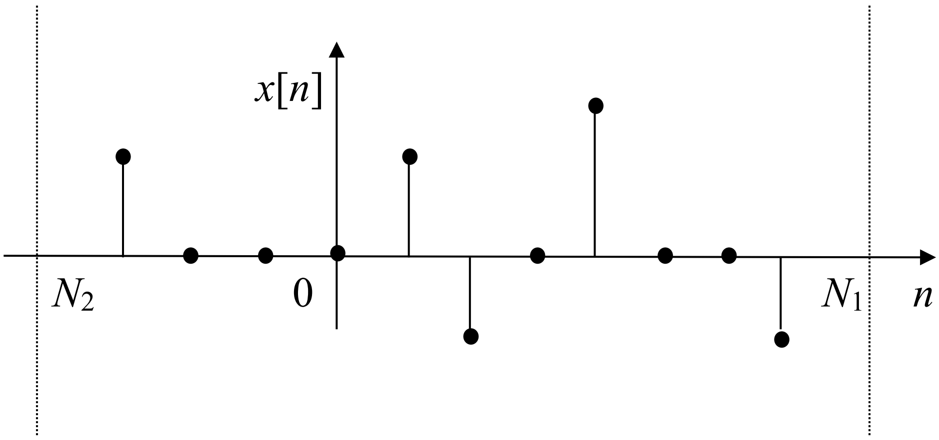


Figure 3.12: An example of finite duration discrete-time sequences.

Equation (3.7) is a special form of convolution. Generally, the convolution of two sequences $x[n]$ and $y[n]$ is defined as

$$\boxed{\begin{aligned} x[n] * y[n] &= \sum_{k=-\infty}^{\infty} x[k]y[n-k] \\ \textit{convolution} \qquad &= \sum_{k=-\infty}^{\infty} x[n-k]y[k] \end{aligned}} \qquad (3.8)$$

Note that, convolution is commutative :

i.e. $x[n] * y[n] = y[n] * x[n]$

3.5.2 The impulse Response of a LTI system

For example consider the discrete-time system, H, shown in Figure 3.13.

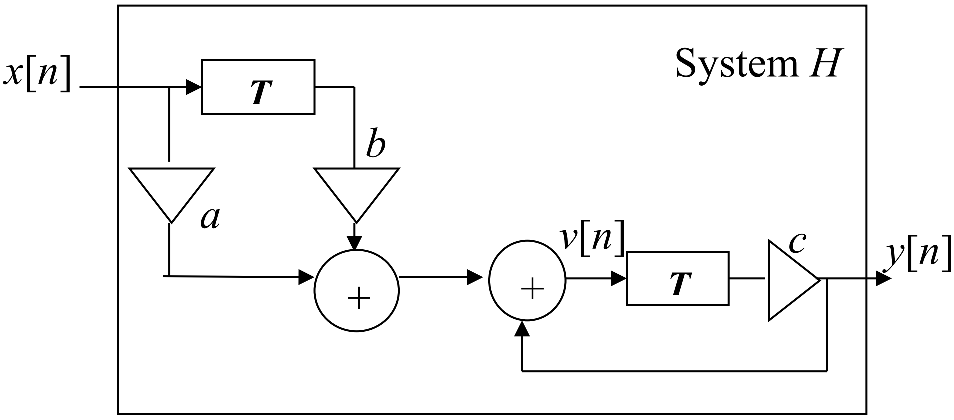


Figure 3.13: An example of discrete-time system, whose input and output are represented by x[n] and y[n], respectively.

Difference equation for the system H:

$$v[n] = ax[n] + bx[n-1] + y[n]$$

$$y[n] = cv[n-1] \qquad (3.9)$$

From the above difference equations, *y[n]* can be determined for a given input.

Let $x[n] = \delta[n]$ ***Unit impulse***

Assume $v[n] = 0$ for $n \leq 0 \Rightarrow y[n]$ is also initially zero for $n \leq 0$. Substituting $n = 0,1,2,...$ in equation (3.9), we obtain

$n = 0$ $v[0]=ax[0]+bx[-1]+y[0]=a\cdot 1+b\cdot 0+0=a$
$\Rightarrow y[1] = a\cdot c$

$n = 1$ $v[1] = ax[1] + bx[0] + y[1] = a\cdot 0 + b\cdot 1 + a\cdot c = b+ac$
$\Rightarrow y[2] = cv[1] = c(b+ac)$

$n = 2$ $\quad v[2] = ax[2] + bx[1] + y[2] = 0 + 0 + c(b+ac)$

$\quad\quad\Rightarrow y[3] = cv[2] = c^2(b+ac)$

...

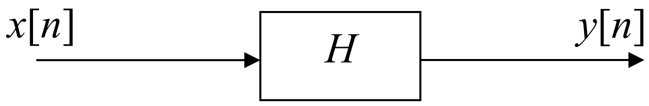
$n = n - 1$ $\quad v[n\text{-}1] = bc^{n-2} + ac^{n-1}$

$\quad\quad\therefore y[n] = c\, v[n\text{-}1] = bc^{n-1} + ac^{n}$

$$\therefore y[n]\Big|_{x[n]=\delta[n]} = h[n] = bc^{n-1}u(n-2) + ac^{n}u(n-1)$$

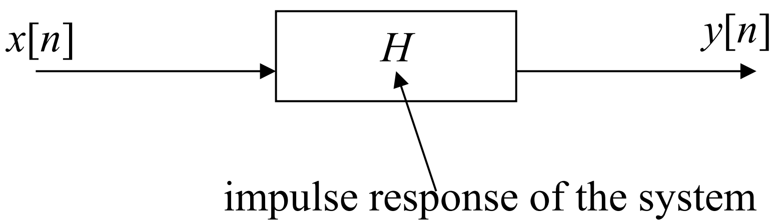
Impulse response

The response $y[n] \stackrel{\Delta}{=} h[n]$ to an impulse excitation ($x[n] = \delta[n]$) is known as the impulse response and it is a very important characteristic of a discrete system.



If $x[n] = \delta[n]$, then $y[n] = h[n]$. The output tells us the system behaviour as the system is being hit by all input frequencies. $h[n]$ completely characterizes the system.The response to an arbitrary input signal $x[n]$ is the convolution of $x[n]$ with the impulse response of the system.

System H:



$$y[n] = x[n] * h[n] \qquad (3.10)$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$
$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \qquad (3.11)$$

where $h[n]=H\{\delta[n]\}$.

3.5.3 Finite Impulse Response (FIR) System

If the impulse response of a LTI system is of finite duration, the system is said to be Finite Impulse Response (FIR).

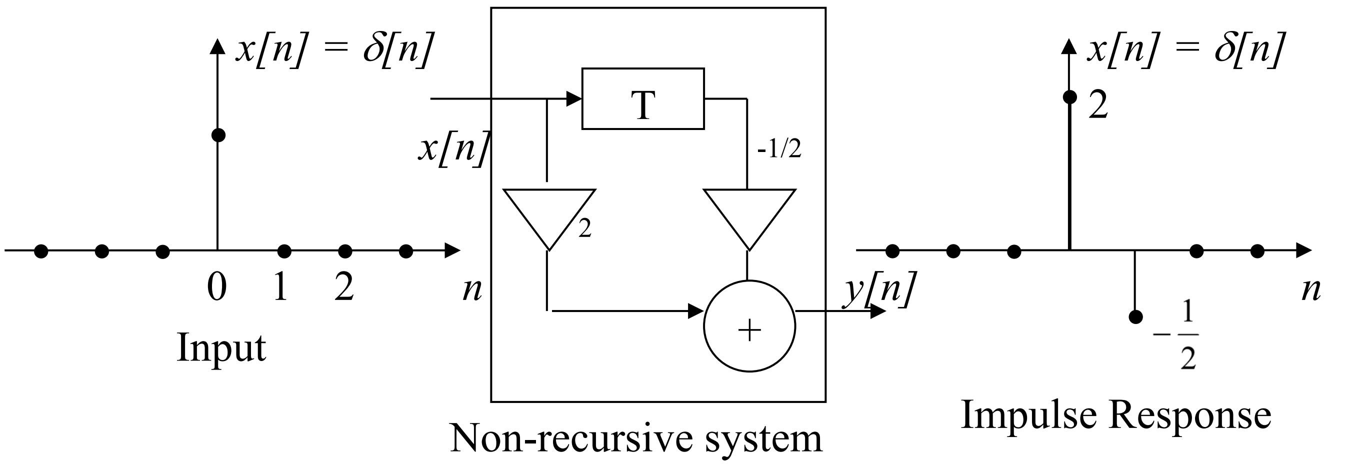


Figure 3.14: An example of LTI systems with finite impulse response.

$x[n] = \delta[n] y[n] = 2x[n] - 0.5x[n-1]$

$n = 0$	$y[0] = h[0] = 2x[0] - 0.5x[-1] = 2$
$n = 1$	$y[1] = h[1] = 2x[1] - 0.5x[0] = -0.5$
$n = 2$	$y[2] = h[2] = 2x[2] - 0.5x[1] = 0$
⋮	⋮
$n = n$	0

3.5.4 Infinite Impulse response (IIR) system

If the impulse response of a linear time-invariant system is of infinite duration, the system is said to be an Infinite Impulse Response (IIR) system.

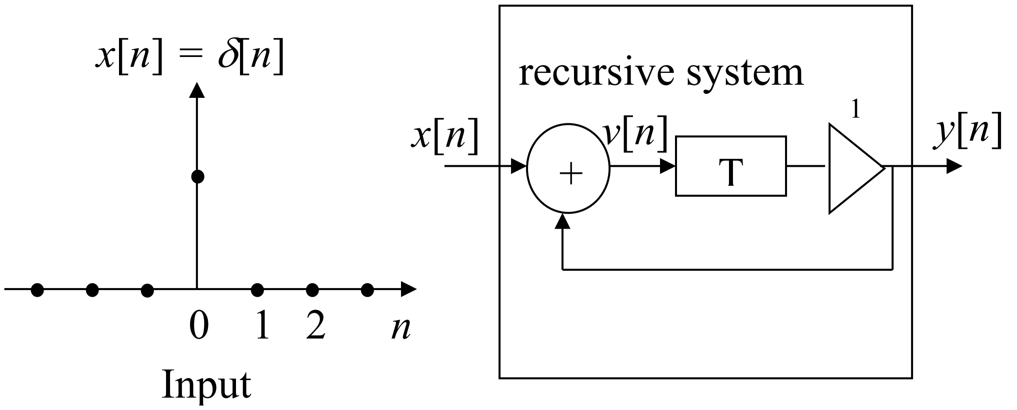


Figure 3.15: An example of discrete systems with infinite impulse response.

$$v[n] = x[n] + y[n]$$
$$y[n] = 1.v[n-1]$$

If $x[n] = \delta[n]$, calculate $h[n]$ for n=0,1,2,...

Example:

Find the impulse response $h[n]$ of the following first-order recursive system.

$$y[n] = \begin{cases} ay[n-1] + x[n] & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

To find $h[n]$, we let $x[n] = \delta[n]$ and apply the zero initial condition.

$n = 0, y[0] = h[0] = ay[-1] + \delta[0] = 1$

$n = 1, y[1] = h[1] = ay[0] + \delta[1] = a$

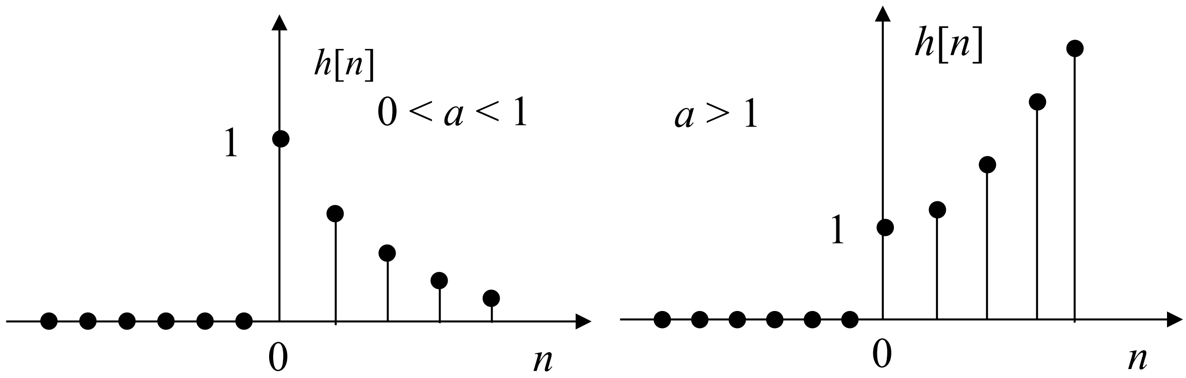
$n = 2, y[2] = h[2] = ay[1] + \delta[2] = a^2$

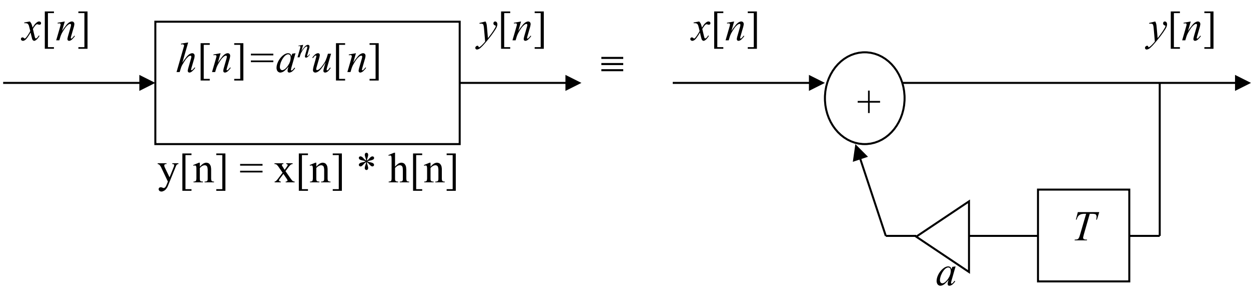
$\vdots$

$n = n, y[n] = h[n] = a^n$ for $n \geq 0$

$y[n] = h[n] = 0$ for $n < 0$, because $\delta[n]$ is zero for $n < 0$ and $y[-1] = 0$.

Hence, $h[n] = a^n u[n]$ for all n





Example:

Find the impulse response $h[n]$ of the following fourth order non-recursive system.

$$y[n] = a_0[n] + a_1 x[n-1] + a_2 x[n-2] + a_3 x[n-3] + a_4 x[n-4]$$

To find $h[n]$, we let $x[n] = \delta[n]$.

$n=0 \Rightarrow h[0] = a_0\delta[0] + a_1\delta[-1] + a_2\delta[-2] + a_3\delta[-3] + a_4\delta[-4] = a_0$

$n=1 \Rightarrow h[1] = a_0\delta[1] + a_1\delta[0] + a_2\delta[-1] + a_3\delta[-2] + a_4\delta[-3] = a_1$

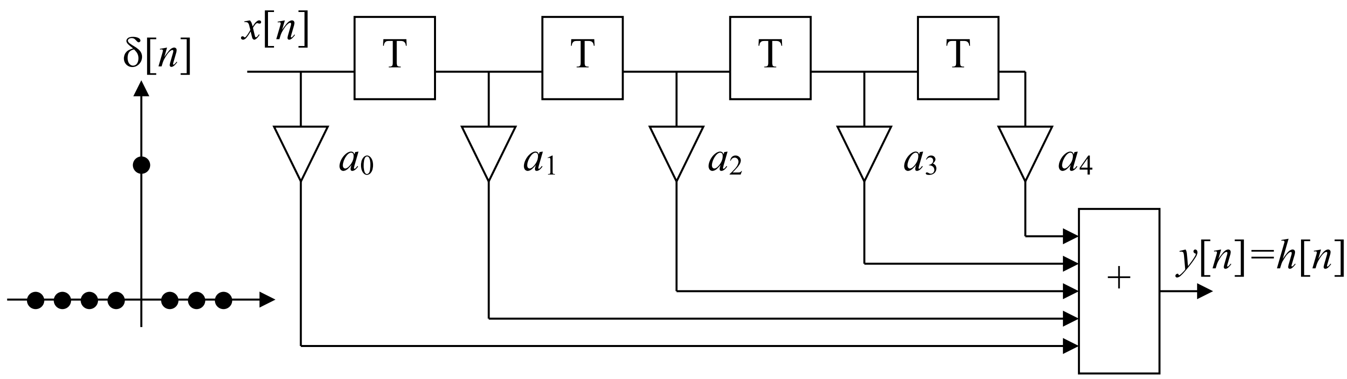
$n=2 \Rightarrow h[2] = a_0\delta[2] + a_1\delta[1] + a_2\delta[0] + a_3\delta[-1] + a_4\delta[-2] = a_2$

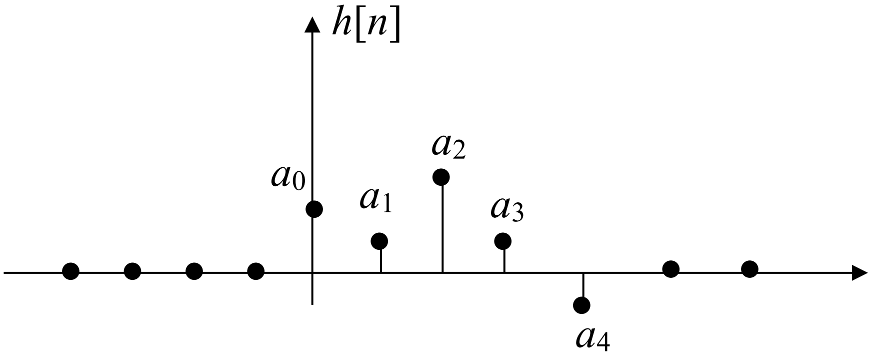
$n=3 \Rightarrow h[3] = a_0\delta[3] + a_1\delta[2] + a_2\delta[1] + a_3\delta[0] + a_4\delta[-1] = a_3$

$n=4 \Rightarrow h[4] = 0 + 0 + 0 + 0 + a_4\delta[0] = a_4$

$n=5 \Rightarrow h[5] = 0 + 0 + 0 + 0 + a_4\delta[1] = 0$

For $n \geq 5$, $h[n] = 0$, since the nonzero value of $\delta[n]$ has moved out of the memory of this system.



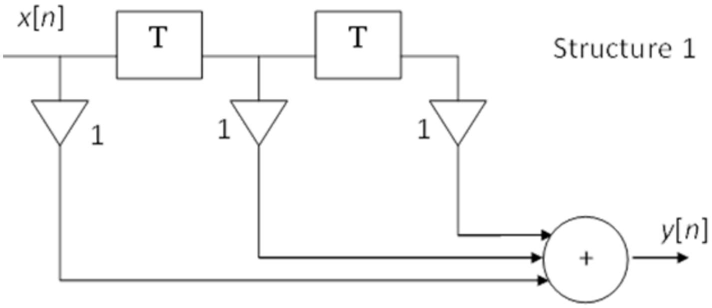


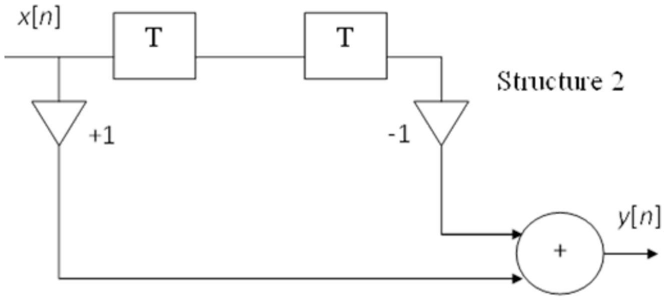
a_0, a_1, a_2, a_3 and a_4 are called coefficients (+ or -) or constants.

Example: Two structures are shown below:

(a) Write the difference equation

(b) Calculate the impulse response





Structure 1: $y[n] = x[n] + x[n\text{-}1] + x[n\text{-}2]$

$h[n]=y[n]|_{x[n]=\delta[n]}$

$h[n] = \delta[n] + \delta[n\text{-}1] + \delta[n\text{-}2]$

$h[0]=1$; $h[1] = 1$; $h[2] = 1$ and $h[n] = 0 \quad n \geq 3$

Structure 2: $y[n] = x[n]\text{-}x[n\text{-}2]$

$h[n] = \delta[n] - \delta[n\text{-}2]$

$h[0]=1$; $h[1] = 0$; $h[2] = -1$ and $h[n] = 0 \quad n \geq 3$

Exercise:

A difference equation for a particular filter is given by

y(n) = 0.12 x(n) – 0.1 x(n-2) + 0.82 x(n-3) – 0.1 x(n-4) + 0.12 x(n-6)

Find the impulse response of the above filter

Ans: h[n]=[0.12 0 -0.1 0.82 -0.1 0 0.12]

3.5.5 Convolution

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

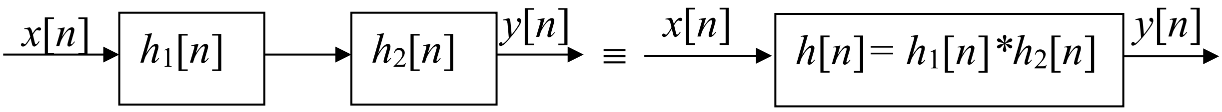
$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

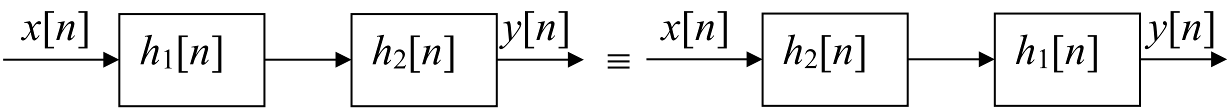
Commutative Law:

$$x[n] * h[n] = h[n] * x[n]$$

Associative Law:

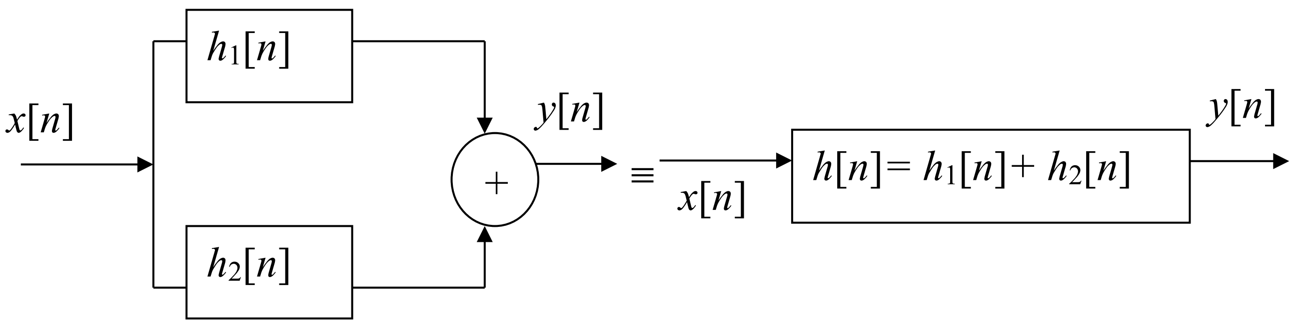
$$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$$





Distributive Law:

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$



Convolution of Finite Sequence

The convolution of two finite-length sequences is also of finite length.

Find the convolution of $h[n]$ = [2, 5, 0, 4] and $x[n]$ = [4, 1, 3]; i.e., $y[n] = x[n] * h[n]$. Assume that both sequences start at n=0. The flipped sequence is $h[-i]$ = [4, 0, 5, 2]. Line up flipped sequence below x[n] to begin overlap and shift it successively, summing the product sequence to obtain the discrete convolution

0 shift

```
x[n]:          4  1  3
 4  0  5  2             ← h[-i]
y[0] = Sum of products = 8
```

1 shift

```
x[n]:          4  1  3
      4  0  5  2        ← h[1-i]
y[1] = (5x4)+(2x1) = 22
```

2 shifts

```
x[n]:          4  1  3
            4  0  5  2  ← h[2-i]
y[2] = 0 + 5 + 6 = 11
```

3 shifts

```
x[n]:          4  1  3
               4  0  5  2 ← h[3-i]
y[3] = 16 + 0 + 15 = 31
```

4 shifts

```
x[n]:       4  1  3
               4  0  5  2 ← h[4-i]
y[4] = 4 + 0 = 4
```

5 shifts

```
x[n]:  4  1  3
               4  0  5  2 ← h[5-i]
y[5] = 12
```

$$\therefore\ y[n] = x[n] * h[n] = [8 \quad 22 \quad 11 \quad 31 \quad 4 \quad 12]$$

Note: Length of $y[n]$ = Length of $x[n]$ + Length of $h[n]$ - 1
6 = 3 + 4 - 1

Exercise:
Compute the convolution $y[n]$ = $x_1[n] * x_2[n]$ of the digital signals given by

$$x_1[n] = [1, -2, 1]$$

$$x_2[n] = \begin{cases} 1, & \text{for } 0 \le n \le 5 \\ 0, & \text{elsewhere} \end{cases}$$

3.6 Stability of Linear Time-Invariant Systems

An LTD system is stable if, and only if, the stability factor denoted by *S*, and defined by

$$\boxed{S = \sum_{k=-\infty}^{\infty} |\, h[k]\,| < \infty \qquad (3.12)}$$

is finite.

Let $x[n]$ be a bounded input sequence {i.e. $|\, x[n]| < M$ for all n, where M is a finite number}. We must show that the output is bounded when S is finite. To this end, we work again with the convolution formula.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

If we take the absolute value of both sides of the above equation, we obtain

$$|\, y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

Now, the absolute value of the sum of terms is always less than or equal to the sum of the absolute values of the terms

$$|\, y[n]| \le \sum_{k=-\infty}^{\infty} |\, h[k]| \quad |\, x[n-k]|$$

Since the input values are bounded, say by M, we have for all n:

$$|\, y[n]| \le \quad M \sum_{k=-\infty}^{\infty} |\, h[k]| \quad \le MS$$

Hence, since both M and S are finite, the output is also bounded. ie, a LTD system is stable if its impulse response is absolutely summable.

Example: Check the stability of the first-order recursive system shown below:

$$y[n] = ay[n-1] + x[n]$$

The impulse response of this system is:

$$h[n] = a^n u[n] \quad \text{for all formula}$$

$$S = \sum_{k=-\infty}^{\infty} |h[n]| \quad = \sum_{k=-\infty}^{\infty} |a|^n$$

It is obvious that S is unbounded for $|a| \geq 1$, since then each term in the series is ≥ 1.

For $|a| < 1$, we can apply the infinite geometric sum formula, to find

$$S = \frac{1}{1-|a|} \quad \text{for } |a| < 1$$

Since S is finite for $|a| < 1$, the system is stable.

Exercise:

For the discrete time system given below, check if it is a linear time-invariant and BIBO stable. Assume $y(-1) = 0$.

$$y(n) = ny(n-1) + x(n) \qquad n \geq 0$$

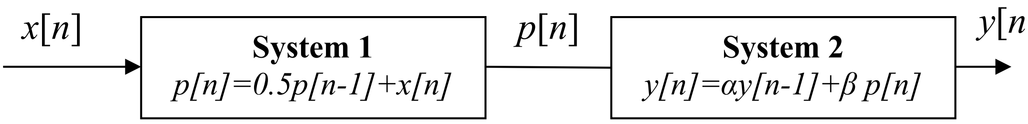
CHAPTER 3: PROBLEM SHEET 3

Q1) Draw the block diagrams of the following system in both Direct form I and Direct Form II.

$$w[n] = -0.5w[n-1] + 7x[n]$$
$$y[n] = 2w[n] - 4w[n-1]$$

(**Note:** $x[n]$ is the input and $y[n]$ is the output)

Q2) Consider the cascade of the following two systems as depicted below:

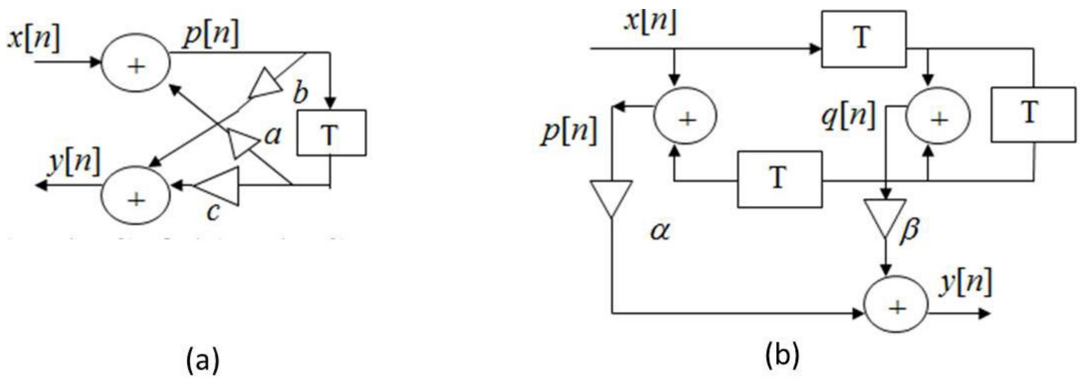


The difference equation relating $x[n]$ of $y[n]$ is:

$$y[n] = x[n] + \tfrac{3}{4}y[n-1] - \tfrac{1}{8}y[n-2]$$

Determine α and β. Ans: α=1/4 and β=1

Q3) For the block diagram realisations given below, develop the relation between $y[n]$ and $x[n]$ in each case.



Ans: a) *y(n)-ay(n-1)=bx(n)+cx(n-1);* b) *y(n)=αx(n)+βx(n-1)+βx(n-2)+αx(n-3)*

Q4) Draw a system implementation for each of the following difference equations:

a) $2y[n] + y[n-1] - 4y[n-3] = x[n] + 3x[n-5]$

b) $y[n] = x[n] - x[n-N]$

c) $y[n] = a_0x[n] + a_1x[n-1] + a_2x[n-2] - b_1y[n-1] - b_2y[n-2]$

Q5) A difference equation for a particular filter is given by

$$y[n] = 0.1x[n] - 0.1x[n-2] + 0.8x[n-3] + 0.1x[n-4] + 0.6x[n-6]$$

Find the impulse response of the above filter.

Ans: h[n]={0.1, 0, -0.1, 0.8, 0.1, 0, 0.6, 0, 0, 0...}

Q6) Show that the convolution of the two infinite duration sequences

$$p[n] = a^n u[n] \quad \text{and} \quad q[n] = b^n u[n]$$

for all n, where $u[n]$ is the unit step function and $a \neq b$, is given by

$$y[n] = \frac{a^{n+1} - b^{n+1}}{a - b}$$

Q7) Consider the system described by the difference equation

$$y[n] = ay[n-1] + bx[n]$$

Determine 'b' in terms of 'a' so that $\sum_{n=-\infty}^{\infty} h[n] = 1$ Ans: b=1-a

Q8) For each of the following systems, determine whether or not the system is (i) linear and (ii) time-invariant

a) $y[n] = \cos(x[n])$ Ans: Non-linear; Time invariant

b) $y[n] = x[n].\cos(0.2\pi n)$ Ans: Linear; Time variant

c) $y[n] = x[n] - x[n-1]$ Ans: Linear; Time invariant

d) $y[n] = |x[n]|$ Ans: Non-linear; Time invariant

e) $y[n] = x[n] + nx[n+1]$ Ans: linear; Time variant

f) $y[n] = \sum_{k=-\infty}^{n+1} x[k]$ Ans: Linear; Time invariant

g) Show that $y[n] = x[-n]$ is not a time-invariant system.

Q9)

a) Let $x[n]$ = {1 4 0 2} and $h[n]$ = {1 2 1}. Find their convolution (Both sequences start at n=0).

Ans:{1, 6, 9, 6, 4, 2}

b) Let $x[n]$ = {0.5 0.5 0.5} and $h[n]$ = {3 2 1}. Find their convolution (Both sequences start at n=0).

Ans:{1.5, 2.5, 3, 1.5, 0.5}

End of Chapter 3