Chapter 2

Chapter 2: Digital Signal Processing Fundamentals

2.1 Introduction

Digital Signal Processing (DSP) is a rapidly developing technology for scientists and engineers. In the 1990s the digital signal processing revolution started, both in terms of the consumer boom in digital audio, digital telecommunications and the wide used of technology in industry.

Due to the availability of low cost digital signal processors, manufacturers are producing plug-in DSP boards for PCs, together with high-level tools to control these boards. There are many areas where DSP technology is now being used and the current proliferation of such technology will open up further applications.

In the medical field, DSP systems are widely utilized for recording data analysis and the interpretation of ECG signals.

Audiologists and speech therapists are exposed to DSP systems for both testing a person's level of hearing and subsequently DSP hearing aid filtering.

The professional music industry uses spectrum analysers, digital filtering, sampling conversion filters etc and is one of the biggest users and exploiters of DSP technology.

In summary, DSP is applied in the area of control and power systems, biomedical engineering, instrumentation (test and measurement), automotive engineering, telecommunications, mobile communication, speech analysis and synthesis, audio and video processing, seismic, radar and sonar processing and neural computing.

There are many advantages to using DSP techniques for variety of applications, these include:

- High reliability and reproducibility
- Flexibility and programmability
- The absence of component drift problem
- Compressed storage facility

DSP hardware allows for programmable operations. Through software, one can easily modify the signal processing functions to be performed by the hardware. For all these reasons, there has been vast growth in DSP theory & applications over the past decade.

2.2 Overview of a DSP System

An analogue signal processing system is shown in Figure 2.1, in which both the input signal and output signal are in analogue form

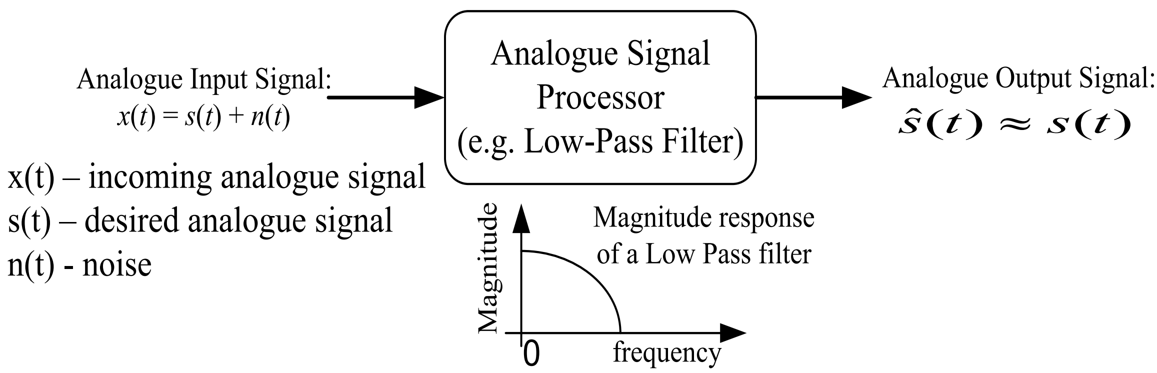


Figure 2.1 A general description of analogue systems whose input and output are in analogue form

A digital signal processing system in Figure 2.1 provides an alternative method for processing the analogue signal.

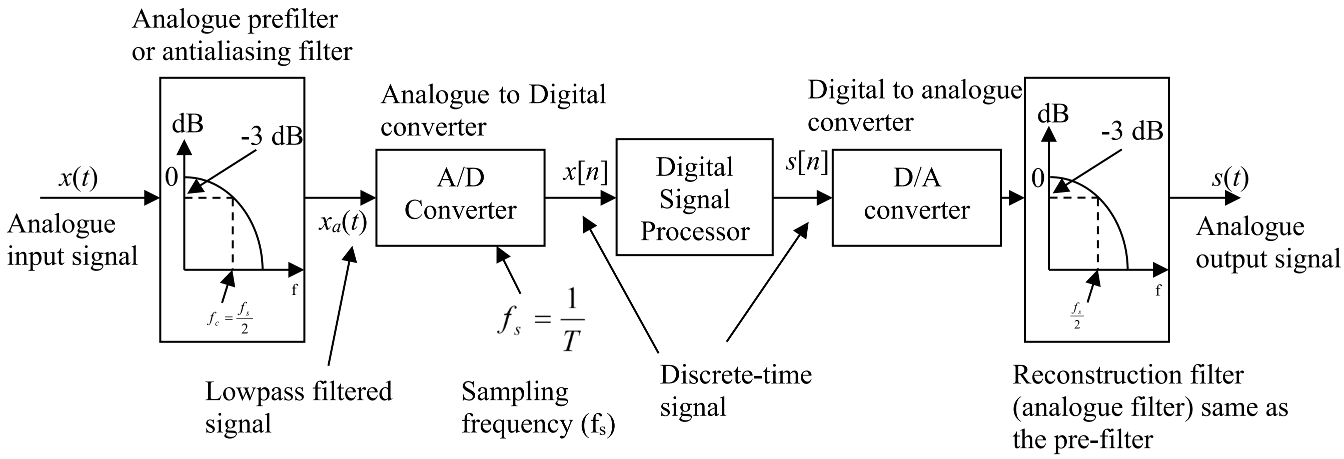


Figure 2.1: A general process of converting analogue signals into digital signals and back to analogue form.

Name	Function
Anti-aliasing filter ($f_c = \frac{f_s}{2}$)	To band-limit the analogue input signal prior to digitisation to reduce **aliasing** (see Section 2.7)
Analogue-to-digital converter	To convert analogue input signal into digital output signal by sampling ($f_s = \frac{1}{T}$)
Digital Signal Processor (heart of the system)	To process the digital signal according to the pre-defined rules
Digital-to-analogue converter	To convert the digital input signal into analogue output signal by interpolating
Reconstruction filter ($f_c = \frac{f_s}{2}$)	To smooth out the output of D/A converter To remove unwanted high frequency components

Table 2.1 Description of blocks contained in a general DSP process

The digital signal processor may implement one of the several DSP algorithms, for example digital filtering (low-pass filter) mapping the digital input signal $x[n]$ into digital output signal $s[n]$.

2.3 Analogue to Digital Conversion Process

Before any DSP algorithm can be performed, the signal must be in a digital form. The A/D conversion process involves the following steps:

- The signal (Band-limited) is first sampled, converting the analogue signal into a discrete-time signal
- The amplitude of each sample is quantised into one of 2^B levels (where B is the number of bits used to represent a sample in the A/D converter)
- The discrete amplitude levels are represented or encoded into distinct binary words each of length B bits.

A practical representation of the A/D conversion process is shown in Figure 2.3.

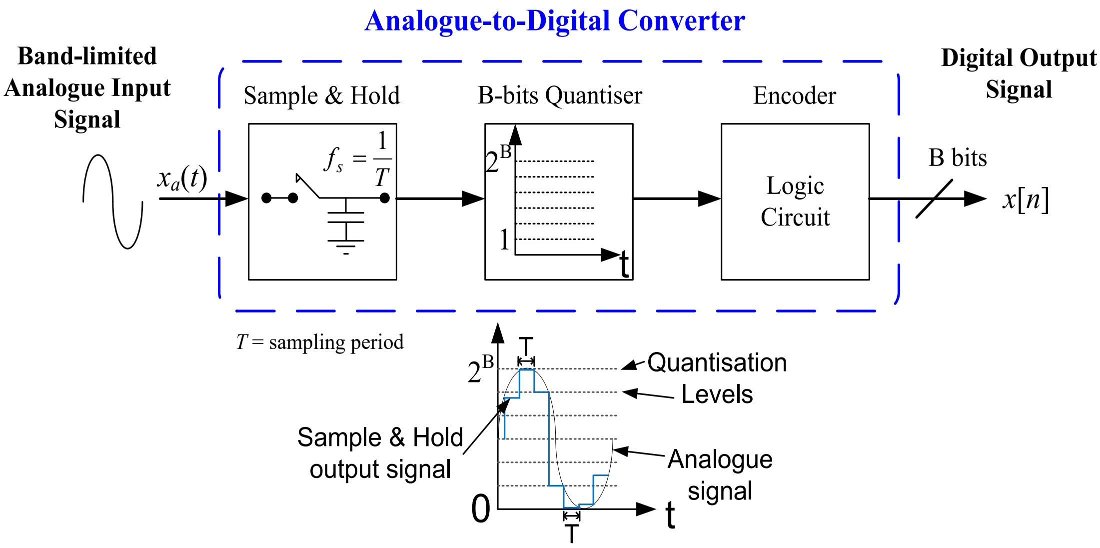
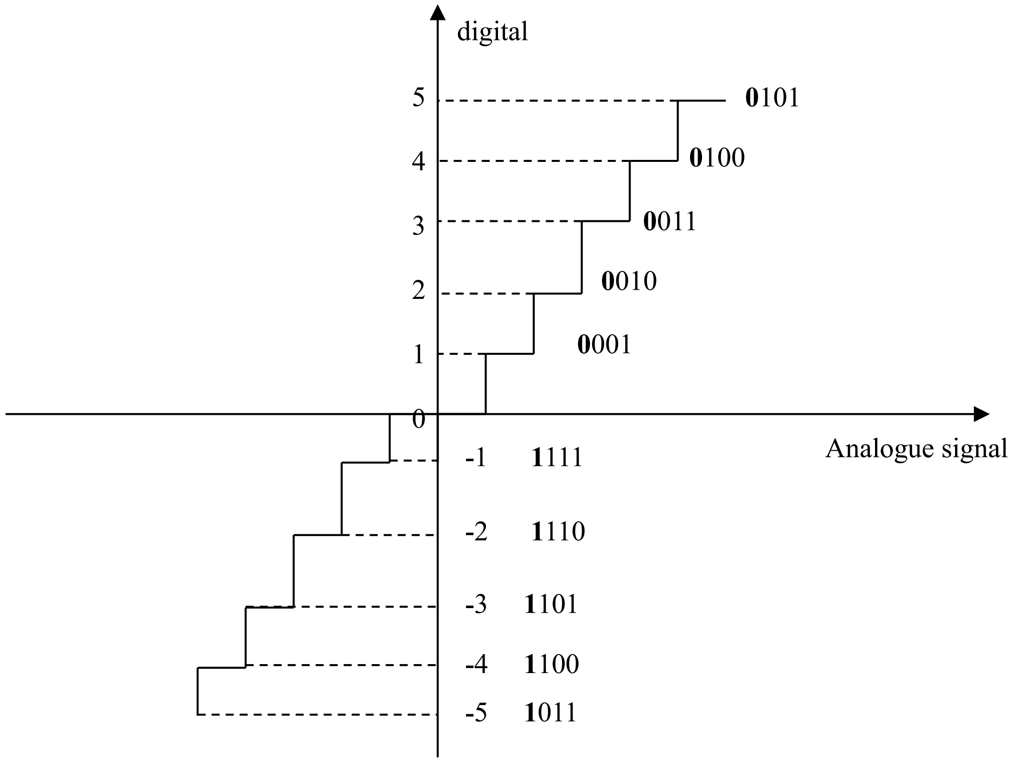


Figure 2.3 Analogue to digital conversion process

Sample and hold (S/H) takes a snapshot of the analogue signal every T sec and then holds that value constant for T secs until the next snapshot is obtained.

<u>Example:</u> 4-bit (B = 4) A/D converter (<u>bipolar</u>)



Input-output characteristic of 4-bit quantiser (linear) (two's complement notation)

2.4 Quantisation and Encoding

Before conversion to digital, the analogue sample is assigned one of 2^B values (see Figure 2.4). This process, termed quantization, introduces an error, which cannot be removed.

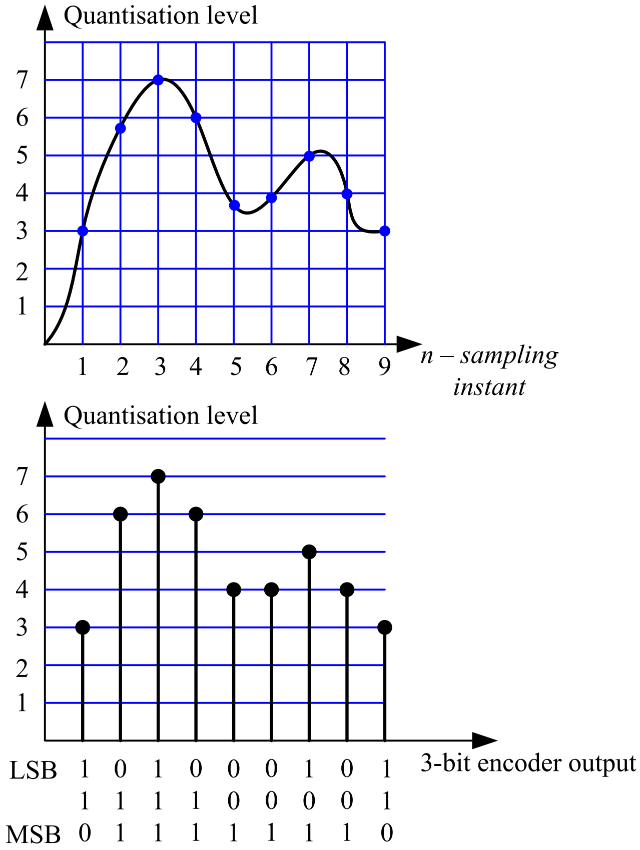
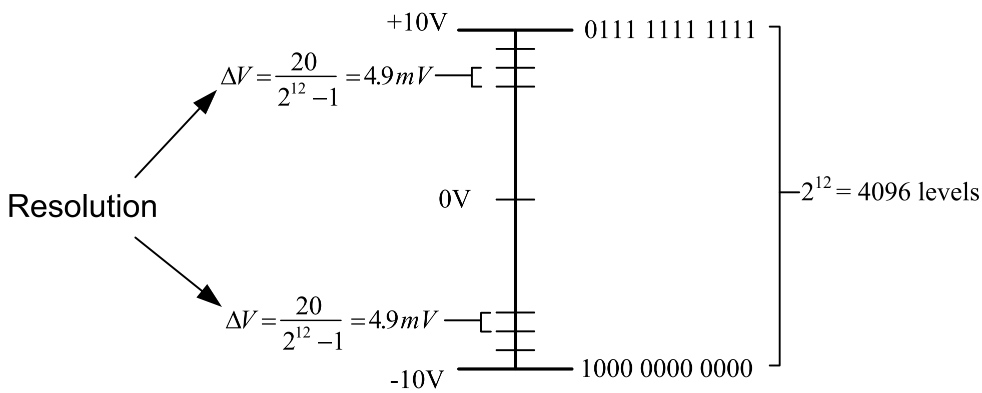


Figure 2.4: Quantisation of discrete-time signals

Example

A 12 bit A/D converter (bipolar) with an input voltage range of ±10V will have a least significant bit (LSB) of $\dfrac{20V}{2^{12}-1} mV = 4.9mV$ (resolution or quantisation step size, ΔV)



For a B-bit A/D converter, the number of quantisation level is 2^B, and the interval between levels, that is known as the quantisation step size or resolution (ΔV) is given by:

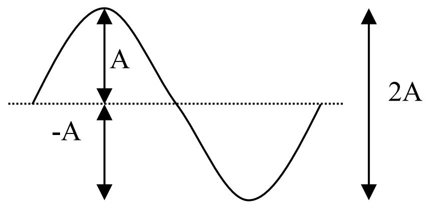
$$\therefore\ \boxed{\Delta V = \frac{V}{2^B - 1} \approx \frac{V}{2^B}}$$

The approximation holds when when B is large (say B > 10 bits)

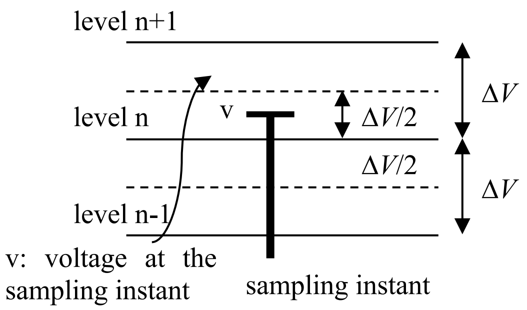
where ΔV = resolution, B = number of bits and V = peak-to-peak amplitude.

For a sine wave input of amplitude A, the quantisation step size becomes

$$\Delta V \approx \frac{2A}{2^B}$$



Quantisation Error (e):



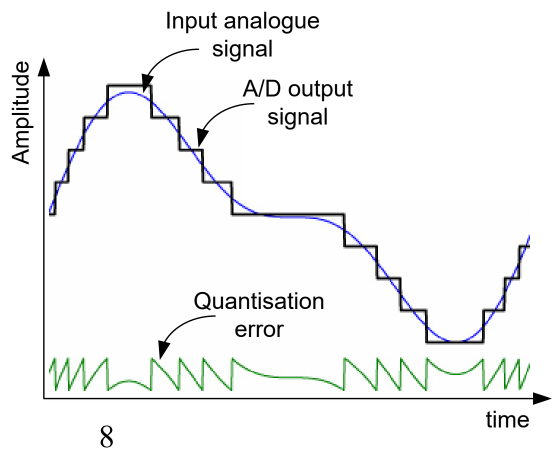
$$\boxed{\text{Quantisation error } (e) = \frac{\Delta V}{2}}$$

For an A/D converter with bipolar signal inputs, the quantisation error (e) is rounded up or down.$\pm\ \Delta V/2$. The quantisation error ($e[n]$) for each sample, is normally assumed to be random and uniformly distributed in the interval $\pm\ \Delta V/2$ with zero mean.

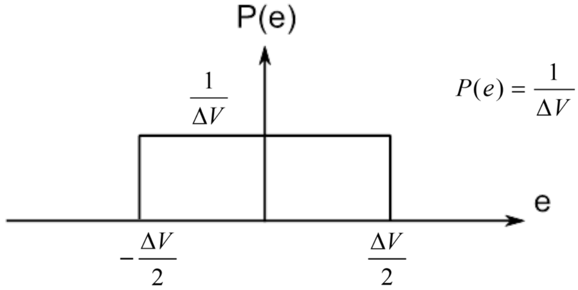
$$e[n] = A - A_q$$

Quantized amplitude

Actual amplitude



The probability density function of the error $P(e)$ has the form as shown below



The quantisation noise power or variance σ_e^2 is hence given by

$$\sigma_e^2 = \int_{-\frac{\Delta V}{2}}^{\frac{\Delta V}{2}} e^2 P(e) de$$

Probability of quantisation error is constant $P(e) = \frac{1}{\Delta V}$

$$= \frac{1}{\Delta V} \int_{-\frac{\Delta V}{2}}^{\frac{\Delta V}{2}} e^2 de = \frac{1}{\Delta V} \left[\frac{e^3}{3} \right]_{-\frac{\Delta V}{2}}^{\frac{\Delta V}{2}}$$

Hence, the quantisation noise power

$$\sigma_e^2 = \frac{\Delta V^2}{12}$$ for uniform quantisation

(Note: Uniform quantisation - all steps (ΔV) are of equal size)

Example

Signal-to-quantisation noise power ratio (SQNR) is defined as

$$SQNR = \frac{P_s}{P_n}$$

signal power

noise power

$$P_s = \frac{1}{N}\sum_{n=1}^{N}\{x[n]\}^2 \quad and \quad P_n = \frac{1}{N}\sum_{n=1}^{N}\{e[n]\}^2$$

$$or \quad \sigma_e^2 = \frac{\Delta V^2}{12}$$

$$SQNR(dB) = 10\log\frac{P_s}{P_n} = 10\log\frac{\sum_{n=1}^{N} x^2[n]}{\sum_{n=1}^{N} e^2[n]}$$

The dynamic range, R, of the signal is defined as

$$R = \{x[n]\}_{\max} - \{x[n]\}_{\min}$$

The quantisation step size of resolution ΔV is defined as

$$\Delta V = \frac{R}{L}$$

number of levels in the quantiser

$$\therefore SQNR(dB) = 10\log\frac{P_s}{\frac{(R/L)^2}{12}} = 10\log\frac{12P_sL^2}{R^2}$$

$$\boxed{SQNR(dB) = 10\log P_s + 20\log L + 10\log 12 - 20\log R}$$

Example: For the sine wave input, the average signal power is $A^2/2$, i.e. $(A/\sqrt{2})^2$ rms value. The signal-to-quantisation noise power ratio (SQNR) in decibels is

$$SQNR = 10\log\left(\frac{\frac{A^2}{2}}{\frac{\Delta V^2}{12}}\right) = 10\log\left(\frac{\frac{A^2}{2}}{\frac{\left(2A/2^B\right)^2}{12}}\right) = 10\log\left(\frac{3\times 2^{2B}}{2}\right)$$

$$\boxed{SQNR = 6.02B + 1.76 \quad \text{dB}}$$

The SQNR increases with the number of bits, B. In many DSP applications, an A/D converter resolution between 12 and 16 bits is adequate.

Number of Bits	Levels	SQNR
3	8	19.7 dB
4	16	25.3 dB
5	32	31.6 dB
6	64	37.7 dB
7	128	43.8 dB

Thus, the signal-to-quantisation noise ratio increases approximately 6dB for each bit.

Exercise: Show that the input signal x(t) to quantisation noise ratio of a linear A/D converter is given by

$$SQNR = 10\log P_s + 10.8 + 20\log L - 20\log R$$

where P_s is the signal power $\left[P_s = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]\right]$; L is the number of quantisation levels and R is the dynamic range of the input signal. Using the above equation, show that for a B-bit quantiser, $SQNR = 6.02B + 1.76\ (dB)$ if $x(t) = A\cos(2\pi f t)$

2.5 Continuous-Time Fourier Transform (FT)

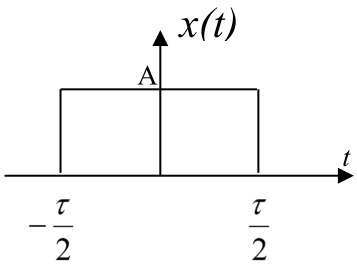
The Fourier transform for non-periodic continuous-time signals is a mathematical transformation to transform signals between time domain and frequency domain, which has many applications in engineering. Most signals of practical importance are non-periodic. The FT pairs is given by

Fourier Transform (FT): $$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Inverse Fourier Transform (IFT): $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

The FT converts the time domain signal, $x(t)$, into its frequency domain representation, $X(\omega)$ and the IFT converts the frequency domain representation, $X(\omega)$, back into the time domain $x(t)$.

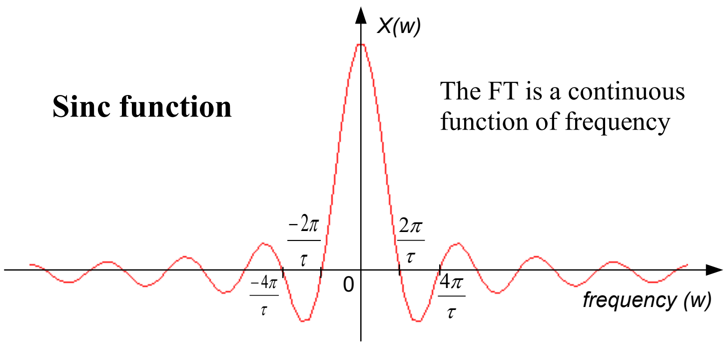
Example: Evaluate the Fourier transform of a rectangular pulse shown below:



$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t)e^{-j\omega t}\,dt = -\frac{A}{j\omega}\left[e^{-\frac{j\omega\tau}{2}} - e^{\frac{j\omega\tau}{2}}\right]$$

$$X(\omega) = A\tau\frac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}$$

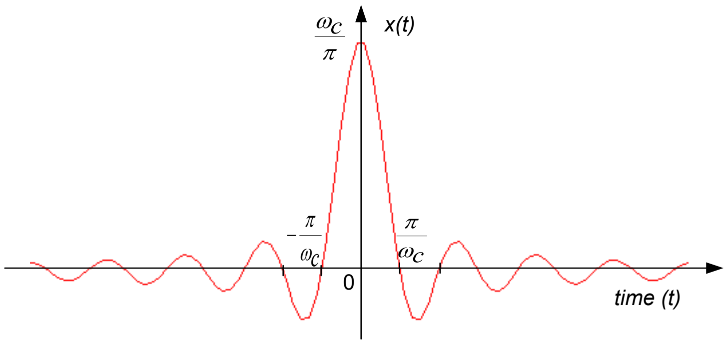
Sinc function



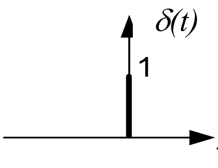
Example: Find the **inverse** Fourier transform of the rectangular spectrum shown below:

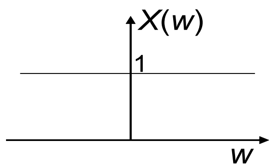
$$x(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} X(\omega)e^{j\omega t}d\omega; \quad x(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1\cdot e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{\pi t}\sin(\omega_C t) = \frac{\omega_C}{\pi}\frac{\sin\omega_C t}{\omega_C t}$$



Example: Find the Fourier Transform of $x(t) = \delta(t)$.





$$X(\omega) = \int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}dt = 1$$

Example:

$$FT\left\{e^{j\omega_1 t}\right\} = \int_{-\infty}^{\infty} e^{j\omega_1 t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{j(\omega_1-\omega)t}dt = 2\pi\delta(\omega_1-\omega)$$

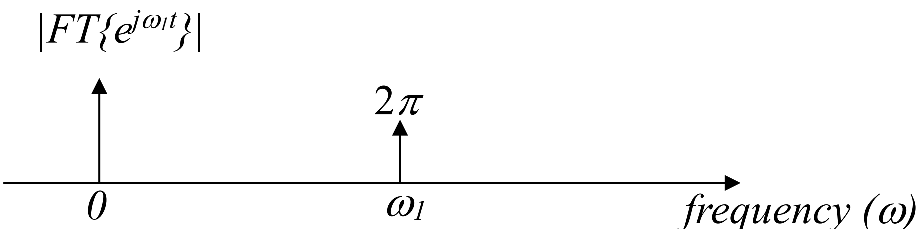
$$\boxed{FT\left\{e^{j\omega_1 t}\right\} = 2\pi\delta(\omega-\omega_1)}$$

Using the properties:

$$\int_{-\infty}^{\infty} e^{j\omega t}dt = 2\pi\delta(\omega)$$

and $\delta(\omega\text{-}\omega_1) = \delta(\omega_1 - \omega)$

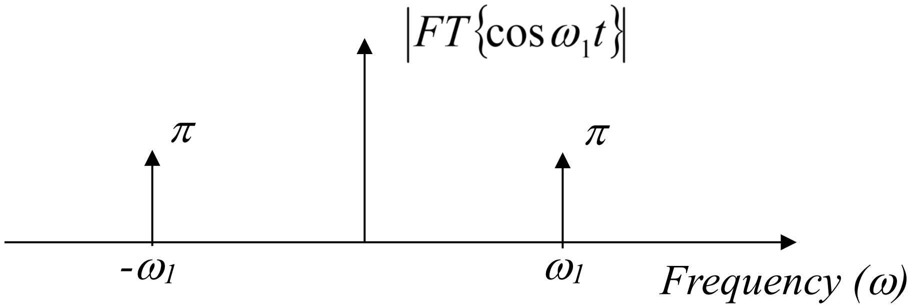
Hence, the magnitude spectrum of $e^{j\omega_1 t}$ is show below:



Example:

$$FT\{\cos\omega_1 t\} = FT\left\{\frac{e^{j\omega_1 t} + e^{-j\omega_1 t}}{2}\right\} = \frac{1}{2}FT\{e^{j\omega_1 t}\} + \frac{1}{2}FT\{e^{-j\omega_1 t}\}$$
$$= \pi\delta(\omega - \omega_1) + \pi\delta(\omega + \omega_1)$$

The magnitude spectrum of $\cos\omega_1 t$ is show below



Exercise:

Find the Fourier Transform of x(t)

$$x(t) = Asin\omega t + Acos2\omega t$$

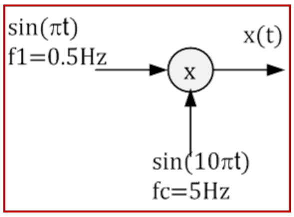
Sketch the magnitude spectrum of $x(t)$

Ans: $X(\omega)=A\pi[\delta(\omega-2\omega_0)+\ \delta(\omega+2\omega_0)-j\delta(\omega-\omega_0)\ +j\delta(\omega+\omega_0)]$

Multiplication of sinusoid

Consider a signal $x(t)$ which is a product of two sinusoids, as given by

$$x(t) = \sin(\pi t)\sin(10\pi t)$$

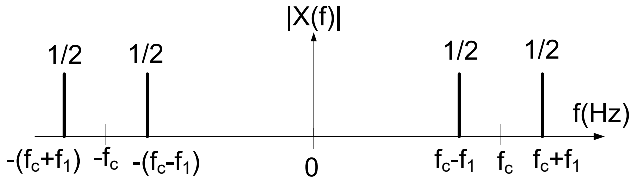


$x(t)$ can be re-written as

$$x(t) = \left(\frac{e^{j10\pi t} - e^{-j10\pi t}}{2j}\right)\left(\frac{e^{j\pi t} - e^{-j\pi t}}{2j}\right) = \frac{1}{4}\left(e^{-j10\pi t} - e^{j10\pi t}\right)\left(e^{-j\pi t} - e^{j\pi t}\right)$$

$$x(t) = \frac{1}{4}e^{j11\pi t} + \frac{1}{4}e^{-j11\pi t} - \frac{1}{4}e^{j9\pi t} - \frac{1}{4}e^{-j9\pi t}$$

From $x(t)$ it can be seen that there are four spectral compoments at frequencies f_c+f_l= 5.5Hz, f_c-f_l = 4.5Hz, $-(f_c+f_l)$ = -4.5Hz, $-(f_c-f_l)$ =-5.5Hz, as shown below



Magnitude spectrum of x(t)

Exercise:

(a) Sketch the magnitude and phase responses of the following signal.

$$x(t) = \sin(2\pi t) + \cos(3\pi t)$$

(b) Find the Fourier Transform of $\sin\omega_1 t$ and draw the amplitude spectrum of $\sin\omega_1 t$.

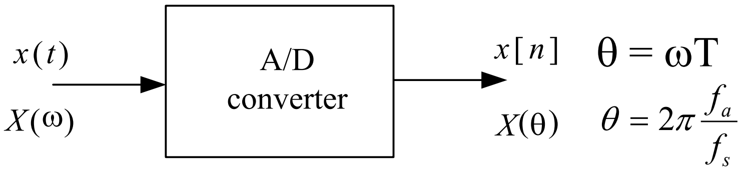
(c) Consider the amplitude modulated signal

$$x(t) = 5\cos(3\pi t)\cos(\pi t), \qquad \text{t - µs}$$

Determine its spectrum (ie. FT) and draw its amplitude spectrum.

2.6 Sampling of continuous-time signal

If a continuous-time signal $x(t)$ is sampled every T seconds, then at the output of the Analogue to Digital Converter, as shown below we obtain a discrete-time signal $x[n]$



where t = time; n = sample number; ω = analogue frequency; θ = digital frequency; $X(\theta)$ = digital spectrum; $X(\omega)$ = analogue spectrum.

The Discrete-Time Fourier Transform (DTFT) pair is given by

Discrete Time Fourier Transform (DTFT):	$X(\theta) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\theta}$
Inverse Discrete Time Fourier Transform (IDTFT):	$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\theta) e^{jn\theta} d\theta$ where $-\pi \le \theta \le \pi$

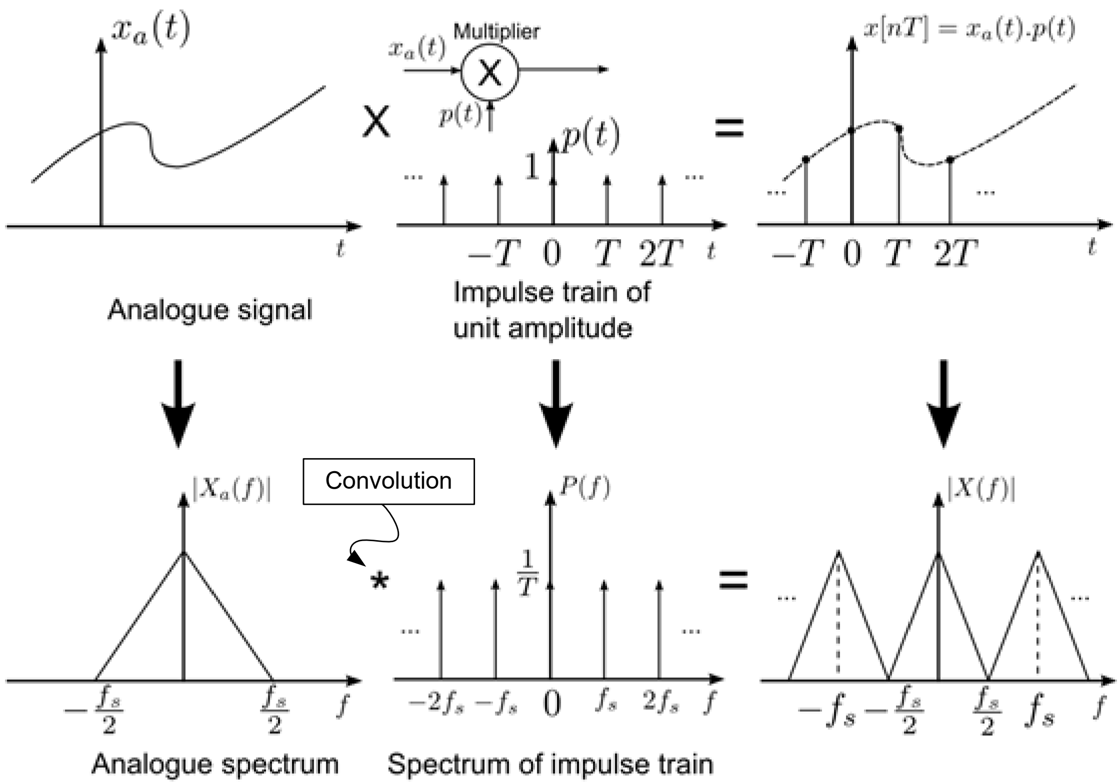
Using Inverse Fourier Transforms in Continuous-time and discrete-time domains, we can show that (see tutorial 2)

$$X(\theta) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega + k\frac{2\pi}{T})$$

It is seen that $X(\theta)$ is periodic with period 2π. The digital spectrum is repetition of the bandlimited analogue spectrum.

2.6.1 The Ideal Sampling Operation

An analogue signal multiplied by a periodic impulse train results in a train of impulses that match the values of the analogue signal at the sampling instants.



Multiplication of the analogue signal and the ideal impulse train results in the convolution of their respective spectra.

The spectrum of the sampled signal $x[n]$ thus consists of replicas of $X_a(f)$ at multiples of the sampling rate f_s ($f_s = \frac{1}{T}$ or $w_s = \frac{2\pi}{T}$).

<u>Exercise:</u> Find the Discrete Time Fourier Transform of x[n]

x[n]=[1 0 0 1]

Ans: $X(\theta)=(2\cos 2\theta)e^{-j2\theta}$

2.7 Aliasing

Aliasing arises when a continuous-time signal is sampled at a rate that is insufficient to capture the changes in the signal. If aliasing occurs, the original continuous time signal cannot be recovered. The Nyquist sampling theorem states that the sampling frequency, f_s, should be at least twice the highest frequency, f_c, contained in the signal ($f_s \geq 2f_c$) to avoid aliasing.

Figure 2.5 illustrates the relationship between the digital spectrum $X(\theta)$ and the analogue spectrum $X(\omega)$ for the case $X(\omega) = 0$, $|\omega| \geq \frac{\pi}{T}$ or $|f| \geq \frac{f_s}{2}$.

Case 1: $X(\omega) = 0$, $|\omega| \geq \frac{\pi}{T}$ (sampling theorem holds)

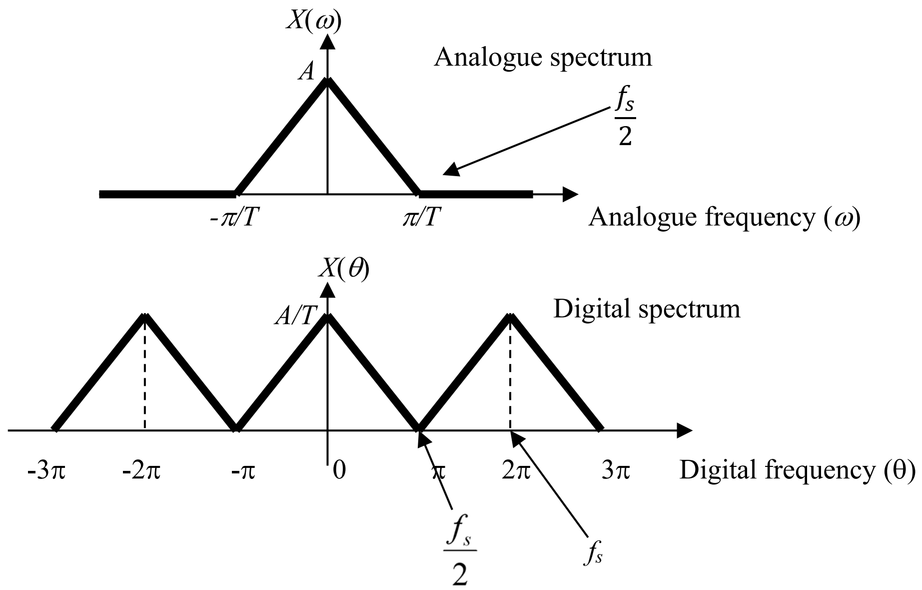


Figure 2.5: Above: Frequency response of an analogue signal. Below: Frequency response of the sampled analogue signal.

Note: $\omega = \frac{\pi}{T}$ corresponds to $\theta = \pi$ (or $f = \frac{f_s}{2}$)

The digital spectrum is the same as the original analogue spectrum and repeats at multiples of the sampling frequency f_s (refer to figure 2.6) as given by:

$$f_k = f_0 + kf_s$$

where k is an integer ($k = \pm 1, \pm 2, \pm 3, \ldots$) and f_0 is the frequency present in the fundamental region of the original analogue spectrum. It is clear that the frequency f_k is outside the fundamental frequency range $-\frac{f_s}{2} \le f_0 \le \frac{f_s}{2}$.

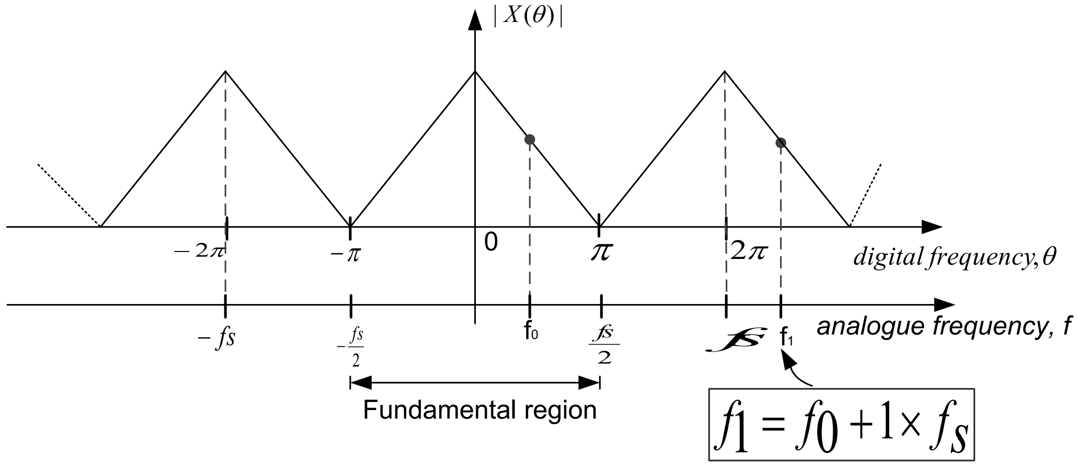


Figure 2.6

Case 2: $X(\omega) \neq 0,\ |\omega| > \frac{\pi}{T}$, but $X(\omega) = 0,\ |\omega| > \frac{3\pi}{2T}$

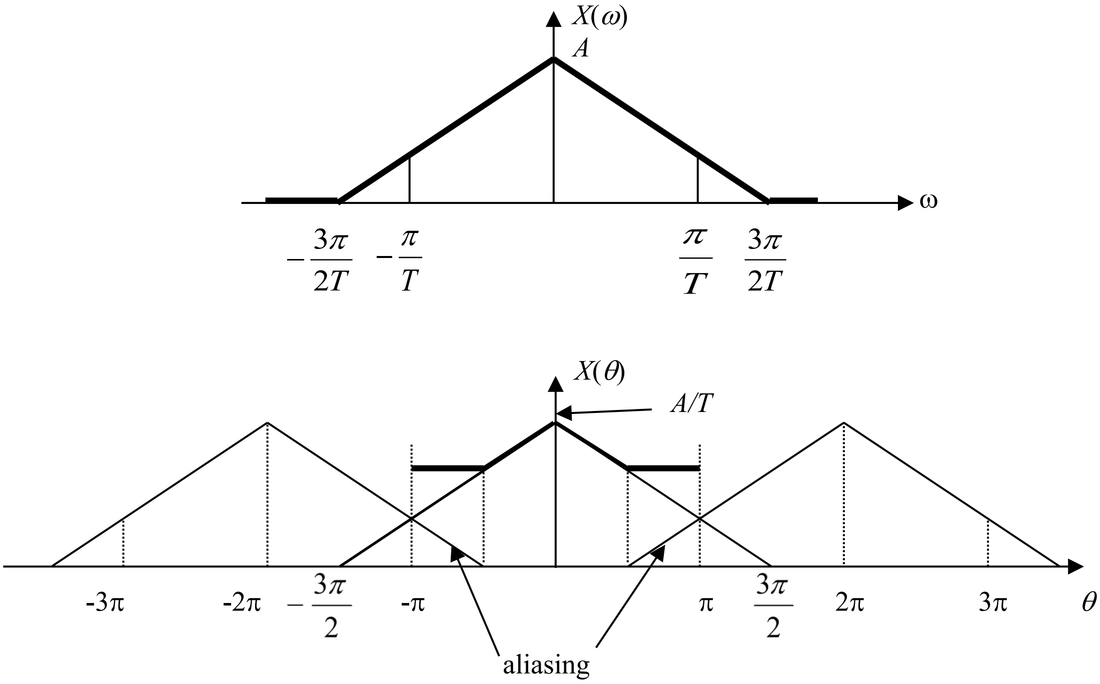


Figure 2.7: Above: Frequency response of an analogue signal whose highest frequency component is larger than the sampling frequency. Below: Frequency response of the sampled analogue signal. The overlapped region represents aliasing.

If the sampling frequency, f_s is not sufficiently high, the spectrum centred on f_s will fold over or alias into the base band frequencies (Figure 2.7). Aliasing can only be avoided if the analogue signal is band limited such that $X(\omega) = 0,\ |\omega| \geq \frac{\pi}{T} \Rightarrow |2\pi f| \geq \frac{\pi}{T} \Rightarrow |f| \geq \frac{f_s}{2}$. This results in the familiar sampling theorem.

Note:

f_s – sampling frequency, f_a – analog frequency

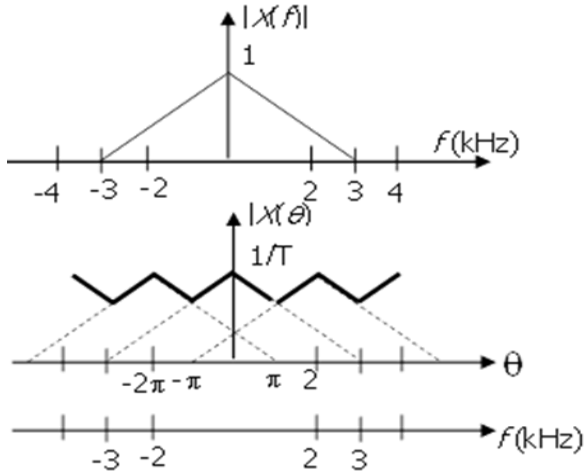
$\frac{f_S}{2}$ corresponds to $\theta = \pi$

$\frac{f_S}{2}$ is the highest frequency that can be represented uniquely with a sampling rate f_s

$\frac{fs}{2}$ is called half the sampling frequency or folding frequency.

Digital frequency, $\theta = \omega T = 2\pi \frac{f_a}{f_s}$

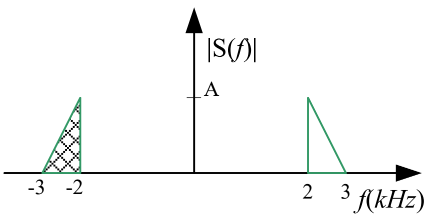
Example: Suppose $x(t)$ has the spectrum $X(f)$ as shown below. Sketch the digital spectrum $|X(\theta)|$ if the sampling frequency f_s = 2 kHz.



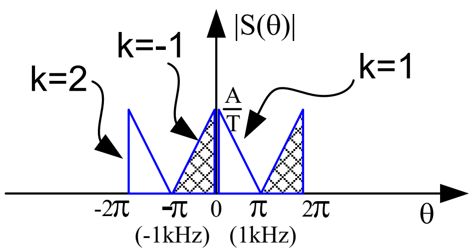
Example: Consider a signal, s(t), with spectrum satisfying the following equation

$$|S(f)| = \begin{cases} 3 - |f| & 2\text{kHz} < |f| < 3\text{kHz} \\ 0 & \text{otherwise} \end{cases}$$

f – frequency in kHz. The signal *s(t)* is sampled uniformly with a sampling frequency of 2kHz. Sketch the digital spectrum $|S(\theta)|$, if it is sampled at a sampling frequency of 2kHz.

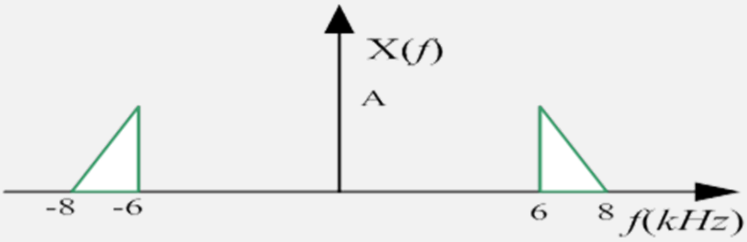


$$f_s = 2\text{kHz}, \quad f_k = f_o + kf_s$$
$$f_o = f_k - kf_s, \quad k = 0, \pm 1, \pm 2, \ldots$$
$$\text{For } k = 1: f_o = 2 - 1(2) = 0; \ f_o = 3 - 1(2) = 1$$
$$\text{For } k = -1: f_o = -2 + 1(2) = 0; \ f_o = -3 + 1(2) = -1$$



Exercise: An analogue signal $x(t)$ with a frequency spectrum shown below is sampled at a rate of 4 kHz. Determine the resulting digital spectrum.

$$X(\theta) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} X(f + kf_S)$$



Example: Consider the analogue signal

$x(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$

What is the Nyquist rate for this signal?

The frequencies present in the signal above are

$f_1 = 25\text{Hz}; f_2 = 150\text{ Hz}; f_3 = 50\text{ Hz}$

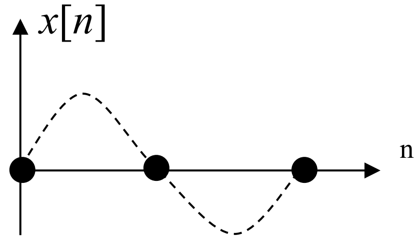
Hence, $f_{\max} = 150\text{ Hz}$

$f_{\text{sampling}} > 2 f_{\max} = 300\text{ Hz}$

The Nyquist rate is $f_N = 2 f_{\max} = 300\text{ Hz}$.

Example : Consider $x(t) = 10\sin(300\pi t)$

$$f_s \geq 2 \times f = 300 \text{ Hz}$$
$$\begin{aligned} x[n] &= 10\sin(300\pi nT) \\ &= 10\sin\left(\frac{300\pi}{f_s}n\right) \\ &= 10\sin(\pi n) \end{aligned}$$



We are sampling the analogue sinusoid at its zero-crossing points and hence we miss the signal completely. The situation will not occur if the sinusoid is offset by some phase (here). In such case we have

$$x(t) = 10\sin(300\pi t + \phi), \text{ where } f_s = 300\text{Hz}.$$
$$\begin{aligned} x[n] &= 10\sin(\pi n + \phi) \\ \therefore \quad &= 10[\sin(\pi n)\cos(\phi) + \cos(\pi n)\sin(\phi)] \quad \text{for n = 0,1,2,..} \\ &= 10\cos(\pi n)\sin(\phi) \end{aligned}$$

Since $\cos(\pi n) = (-1)^n$, $\quad x[n] = (-1)^n 10\sin(\phi)$

If θ = 0 or θ = π, the samples of the sinusoid taken at the Nyquist rate are not all zero.

Example: Consider the analogue signal

$$x(t) = 3\cos(2000\pi t) + 5\sin(6000\pi t) + 10\cos(12000\pi t)$$

(a) What is the Nyquist rate for this signal?

The frequencies existing in the analogue signal are:

f_1 = 1 kHz; f_2 = 3 kHz; f_3 = 6 kHz

Thus f_{max} = 6 kHz and according to the sampling theorem,

$f_s > 2 f_{max}$ = 12 kHz

The Nyquist rate is = 12 kHz.

(b) Assume now that we sample this signal $x(t)$ using a sampling rate f_s = 5 kHz (samples/sec). What is the discrete-time signal obtained after sampling?

First Method:

$f_s = 5000\text{Hz} \Rightarrow \frac{f_s}{2} = 2500$

$x(t) = 3\cos(2\pi \times 1000t) + 5\sin(2\pi \times 3000t) + 10\cos(2\pi \times 6000t)$

$$x[n] = 3\cos\left(2\pi\frac{1000}{5000}n\right) + 5\sin\left(2\pi\frac{3000}{5000}n\right) + 10\cos\left(2\pi\frac{6000}{5000}n\right)$$

$$= 3\cos\left(2\pi\left(\frac{1}{5}\right)n\right) + 5\sin\left(2\pi\frac{3}{5}n\right) + 10\cos\left(2\pi\left(\frac{6}{5}\right)n\right)$$

$$= 3\cos\left(2\pi\left(\frac{1}{5}\right)n\right) + 5\sin\left(2\pi\left(1-\frac{2}{5}\right)n\right) + 10\cos\left(2\pi\left(1+\frac{1}{5}\right)n\right)$$

$$= 3\cos\left(2\pi\left(\frac{1}{5}\right)n\right) + 5\sin\left(-2\pi\frac{2}{5}n\right) + 10\cos\left(2\pi\left(\frac{1}{5}\right)n\right)$$

$$x[n] = 13\cos\left(2\pi\left(\frac{1}{5}\right)n\right) - 5\sin\left(2\pi\left(\frac{2}{5}\right)n\right)$$

Second Method:

$$f_s = 5kHz \Rightarrow \frac{f_s}{2} = 2.5kHz$$

We have $f_k = f_0 + kf_s$

$f_0 = f_k - kf_s$ can be obtained by subtracting from f_k an integer multiple of f_s such that $-\frac{f_s}{2} \le f_0 \le \frac{f_s}{2}$.

The frequency f_1 = 1000 Hz is $< \frac{f_s}{2}$ (= 2500 Hz) and thus it is not affected by aliasing.

However, the other two frequencies f_2 & f_3 are above the folding frequency and they will be changed by the aliasing effect.

$f_2 = f_2 - 1\, f_s = 3000 - 5000 = -2$ kHz

$f_3 = f_3 - 1\, f_s = 6000 - 5000 = 1$ kHz

$$\boxed{x[n] = 3\cos\left(2\pi\left(\frac{1000}{5000}\right)n\right) + 5\sin\left(2\pi\left(-\frac{2000}{5000}\right)n\right) + 10\cos\left(2\pi\left(\frac{1000}{5000}\right)n\right)}$$

This is agreement with the result obtained before.

(c) What is the analogue signal $y(t)$ we can reconstruct from the samples if we use ideal interpolation?

Since only frequency components at 1 kHz and 2 kHz are present in the sampled signal, the analogue signal we can recover is,

$$y(t) = 13\cos(2000\pi t) - 5\sin(4000\pi t)$$

which is obviously different from the original signal $x(t)$.
The distortion of the original analogue signal was caused by the aliasing effect, due to the low sampling rate used.

<u>Exercise</u>: A digital communication link caries binary-coded words representing samples of an input signal

$$x(t) = 5\cos(600\pi t) + 4\cos(1800\pi t)\ .$$

The link is operated at 10,000 bits/s and each input sample is quantised into 1024 different voltage levels.

(i) What are the frequencies in the resulting discrete-time signal x(n)?

(ii) Determine the resulting discrete time signal x(n).

2.8 Digital-to-Analogue Conversion (D/A) – Signal recovery

The D/A conversion process is employed to convert the digital signal into an analogue form after it has been digitally processed. The reason for such conversion may be for example, to generate an audio signal to drive a loudspeaker or to sound an alarm. The D/A process is shown in Figure 2.8. A register is used to buffer the D/A's input to ensure that its output remains the same until the D/A is fed the next digital input.

Note: The inputs to the D/A are series of impulses, while the output of the DAC has a staircase shape as each impulse is held for a time T sec.

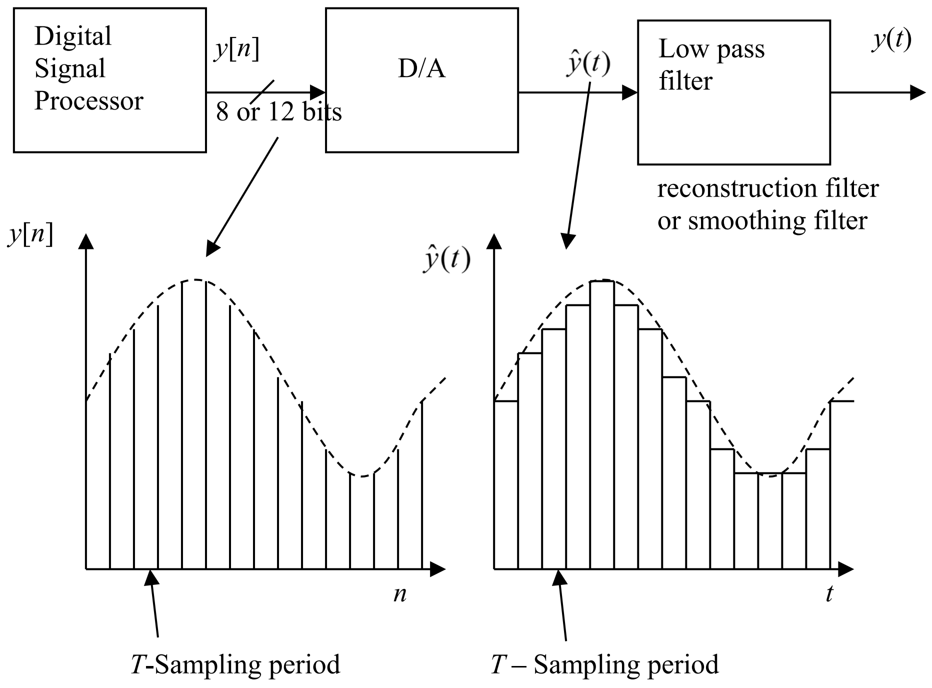


Figure 2.8: Conversion process from digital signals to analogue signals.

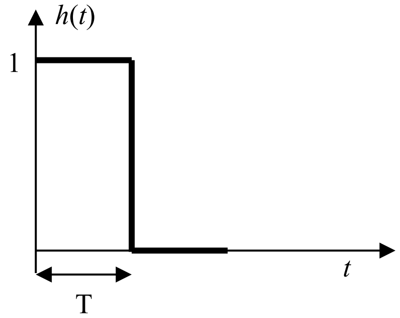
The D/A shown in Figure 2.8 is referred to as a <u>zero-order</u> hold.

By comparing its output $\hat{y}(t)$ and its input $y[n]$, it is evident that for each digital code fed into the D/A, its output is held for a time T. The result is the characteristic staircase shape at the D/A output.

The D/A output approximates the analogue signal by a series of rectangular pulses whose height is equal to the corresponding value of the signal pulse.

Just consider one pulse.

$$h(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & otherwise \end{cases}$$



The corresponding frequency response is

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \int_{0}^{T} e^{-j\omega t}dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^T = \frac{T}{2} \quad e^{-\frac{j\omega T}{2}} \quad \frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}$$

The magnitude of $H(\omega)$ is plotted in Figure 2.9.

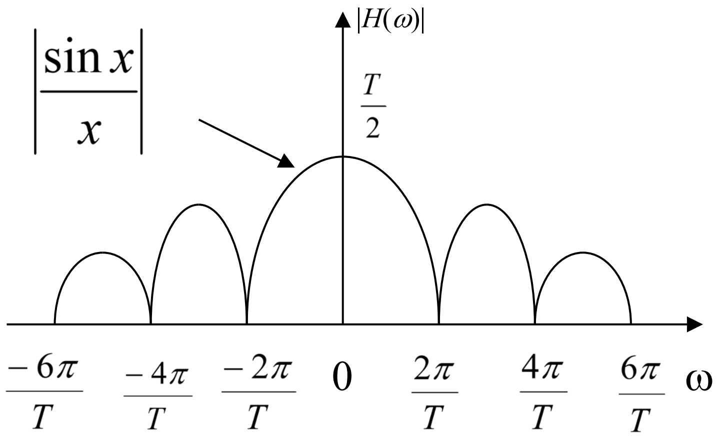
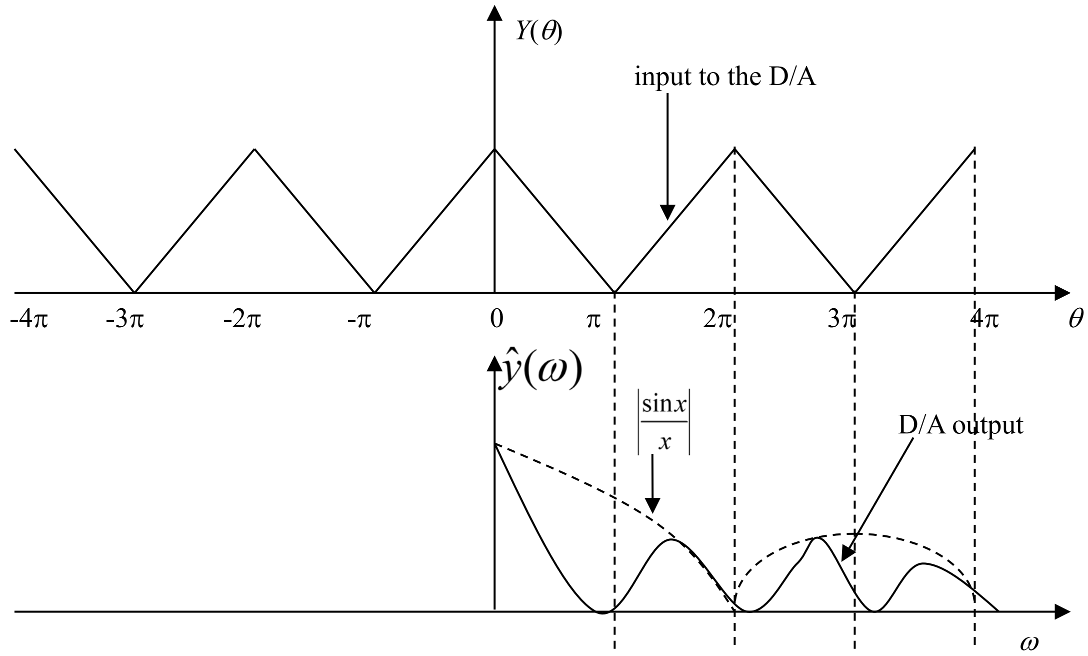


Figure 2.9: Magnitude response of a rectangular pulse.

In the frequency domain, the staircase action of the DAC introduces a type of distortion known as the $\frac{\sin x}{x}$ or aperture distortion, where $x = \frac{\omega T}{2}$.



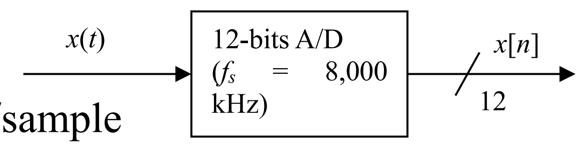
The amplitude of the output signal spectrum is multiplied by the $\frac{\sin x}{x}$ function, which acts like a lowpass filter, with the high frequencies heavily attenuated. The $\frac{\sin x}{x}$ effect is due to the holding action of the DAC and, in signal recovery, introduces an amplitude distortion. For a zero-order hold, the function $\frac{\sin x}{x}$ falls to about 4 dB at half the sampling frequency $\left(\frac{f_s}{2}\right)$ giving an average error of about 36.4%. Aperture error can be eliminated by equalization. In practice this can be achieved by first applying the signal, before converting it to analogue, through a <u>digital filter</u> whose amplitude-frequency response has a $\frac{x}{\sin x}$ shape.

2.8.1 Reconstruction Filter

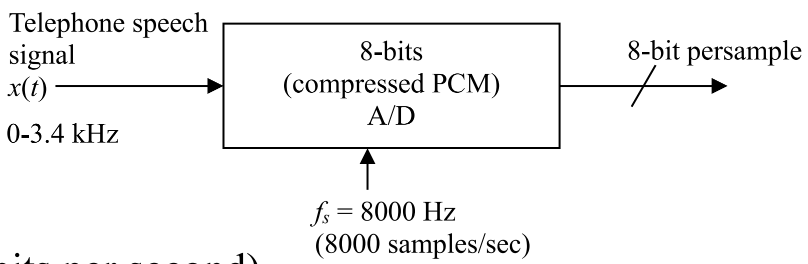
The output of the D/A converter contains unwanted high frequency at multiples of the sampling frequency as well as the desired frequency components. The role of the output filter is to smooth out the steps in the D/A output thereby removing the unwanted high frequency components. In general, the requirements of the anti-imaging filter are similar to those of the anti-aliasing filter.

Note:

(a) bit rate = f_s × no of bits
= 8000 samples/sec × 12 bits/sample
= 96000 bits/sec

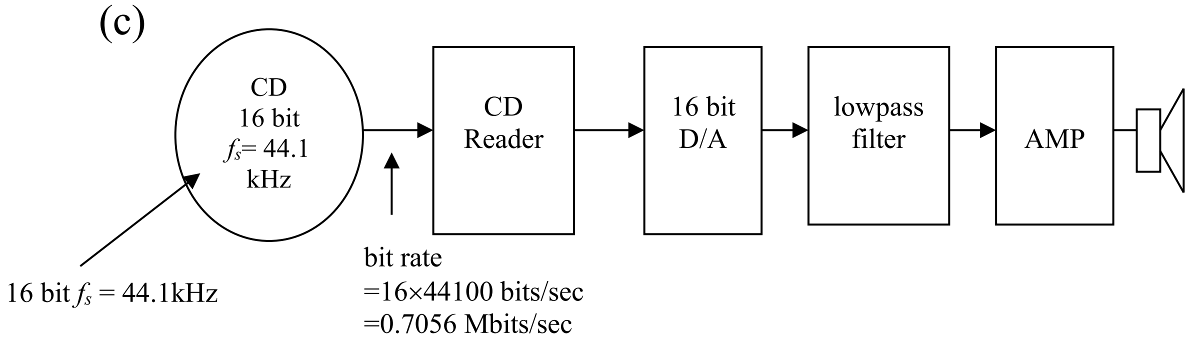


(b) In the case of Pulse Code Modulation (PCM), speech signals are filtered to remove effectively all frequency components above 3.4 kHz and the sampling rate is 8000 samples per sec



Bit rate (bits per second)
= sampling frequency × bits/sample
= 8000 samples/second × 8 bits/sample
= 64,000 bits/sec

(c)



CHAPTER 2: PROBLEM SHEET 2

Q1) For a linear 16 bit A/D converter with an input signal range of ±4V, what is the minimum quantisation error? Ans: $1/2^{14}$

Q2) A sampled signal that varies between -2V and 2V is quantised using B bits. What value of B will ensure that the quantisation noise power is less than 25×10^{-6}? Ans: 8 bits

Q3) A sinusoidal signal with peak-to-peak amplitude of 5V is sampled at 50kHz with uniform quantisation. Find the minimum number of bits for the analogue to digital converter to achieve a SQNR of at least 92dB. State any assumptions made. Ans: $\text{B} = 15$

Q4) Show that the signal to quantisation noise ratio (SQNR) of a linear B-bit analogue-to-digital converter is given by

$$SQNR = 6.02B + 4.77 - 20\log\left(\frac{\text{A}}{\sigma_{\text{sig}}}\right)$$

where the input range of the A/D converter is ±A and the rms value of the input signal is σ_{sig}. Determine the SQNR if B is 16 bits and the input is a signal with an rms value of $\frac{A}{5}$. Ans: $SQNR = 6.02 \times 16 + 4.77 - 20log\frac{A}{A} \times 5$

Q5) An analogue signal $x(t) = \sin(480\pi t)+3\sin(720\pi t)$ is sampled 600 times per second.

(a) Determine the Nyguist sampling rate for $x(t)$.

(b) Determine the folding frequency (or half the sampling frequency).

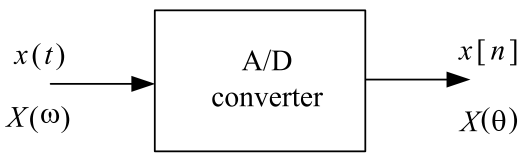
(c) What are the frequencies, in radians, in the resulting discrete time signal $x[n]$?

(d) If $x[n]$ is passed through an ideal D/A converter what is the reconstructed signal $y(t)$?

Q6) An analogue signal $x_a(t) = \cos(2\pi 500t)$ is sampled at a rate of f_s=4kHz. Determine the resulting analogue and digital magnitude spectra.

Q7) Find the Fourier Transform of $x(t) = \delta(t-a)$. *Hence,* show that $\int_{-\infty}^{\infty} e^{j\omega t} dt = 2\pi\delta(\omega)$.

Q8) If an analogue signal $x(t)$ is sampled every T seconds, then at the output of the Analogue to Digital Converter, as shown in a diagram below:



Using Inverse Fourier Transforms in Continuous time and discrete time domains, show that

$$X(\theta) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega + k\frac{2\pi}{T}).$$

End of Chapter 2