Chapter 1

Chapter 1: Signals and Systems

1.1 Introduction

The terms 'signals' and 'systems' are given various interpretations. For example, a **system** is an electric network consisting of resistors, capacitors, inductors and energy sources. **Signals** are various voltages and currents in the network. The signals are thus functions of time and they are related by a set of equations.

Example:

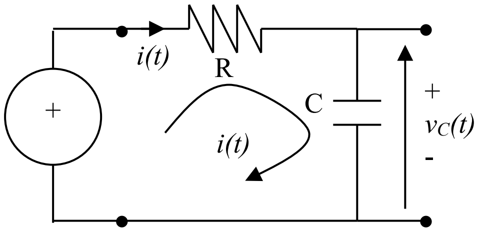


Figure 1.1: An electric circuit

The objective of system analysis is to determine the behaviour of the system subjected to a specific input or excitation. It is often convenient to represent a system schematically by means of a box as shown in Figure 1.2.

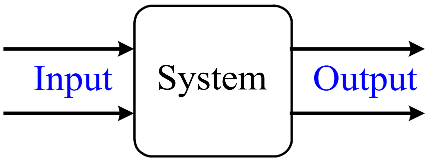


Figure 1.2: General representation of a system.

1.2 Signals

There are two types of signals:

(a) Continuous – time signals

(b) Discrete – time signals

In the case of a continuous-time signal, $x(t)$, the independent variable t is continuous and thus $x(t)$ is defined for all t (see Figure 1.3) where t is a continuos time -independent variable $(-\infty < t < \infty)$

On the other hand, discrete-time signals are defined only at discrete times and consequently the independent variable takes on only a discrete set of values (see Figure 1.3). A discrete-time signal is thus a sequence of numbers where n is a discrete time -independent variable $(n = \ldots -2, -1, 0, 1, 2, \ldots)$

<u>Examples:</u>

- A person's body temperature is a **continuous-time** signal.
- The prices of stocks printed in the daily newspapers are **discrete-time** signals.
- Voltages and currents are usually represented by **continuous-time** signals. They are also represented by **discrete-time** signals if they are specified only at a discrete set of values of t.

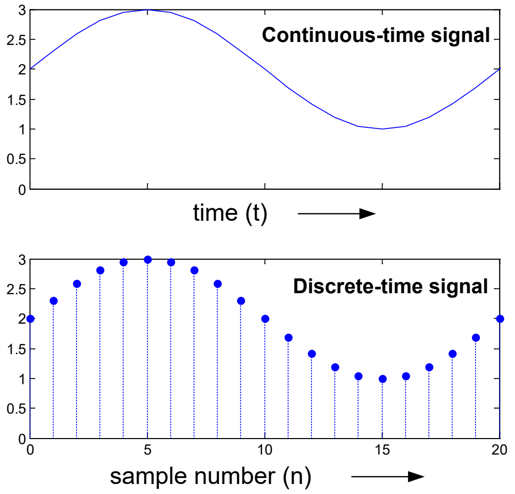


Figure 1.3: An example of continuous-time and discrete-time signals

1.2.1 Sampling

A discrete-time signal is often formed by sampling a continuous -time signal *x(t)*. If the samples are equidistant then

$$\boxed{x[n] = x(t)\big|_{t=nT} = x(nT) \qquad (1.1)}$$

Square brackets [ ] ⇒ Discrete time signals
Round Brackets ( ) ⇒ Continuous signals

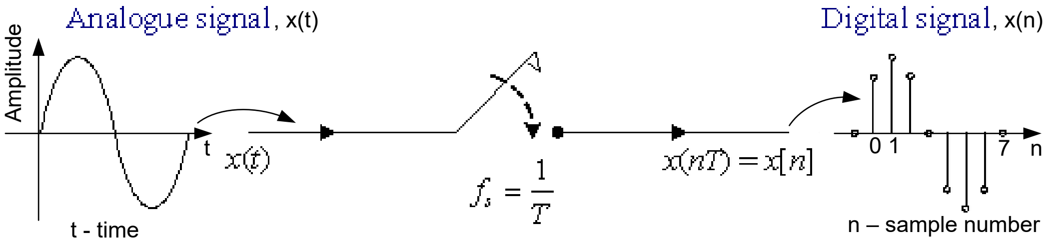


Figure 1.4 An example of acquiring discrete-time signal by sampling continuous-time signal

The constant *T* is the sampling interval or period and $f_s=1/T$ is the sampling frequency.

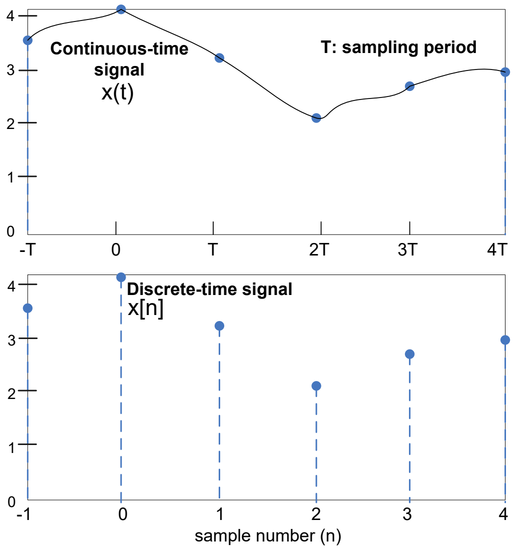


Figure 1.5: An example of acquiring discrete-time signals by sampling continuous-time signals.

$$x[n] = \{ \underset{\substack{\uparrow \\ n=-1}}{3.5}, \ \underset{\substack{\uparrow \\ n=0}}{4}, \ 3.25, \ \underset{\substack{\uparrow \\ n=2}}{2}, \ 2.5, \ \underset{\substack{\uparrow \\ n=4}}{3.0} \}$$

It is important to recognize that $x[n]$ is only defined for integer values of n. It is not correct to think of $x[n]$ as being zero for n not an integer, say n=1.5. $x[n]$ is simply undefined for non-integer values of n.

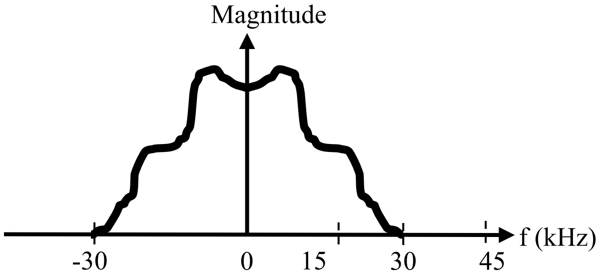
Sampling Theorem

If the highest frequency contained in an analogue signal $x(t)$ is $f_{\max}$ and the signal is sampled at a rate $f_s \geq 2\, f_{\max}$ then $x(t)$ can be exactly recovered from its sample values using an interpolation function.

For example, audio CDs use a sampling rate $f_s = 44.1kHz$ for storage of the digital audio signal. This sampling frequency is slightly more than $2 \times f_{\max}$ [$f_{\max}$ = 20kHz], which is generally accepted upper limit of human hearing and perception of music sounds.

Exercise:

What is the minimum sampling frequency, f_s, required to avoid aliasing when sampling a signal with the following spectrum?

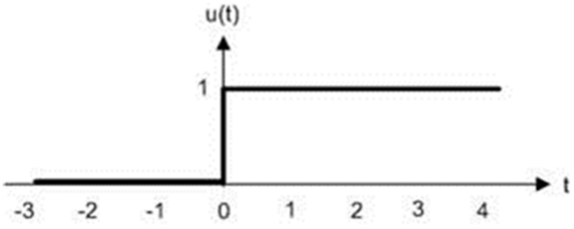


a) 30 kHz
b) 60 kHz
c) 45 kHz
d) none of the above

Unit step function

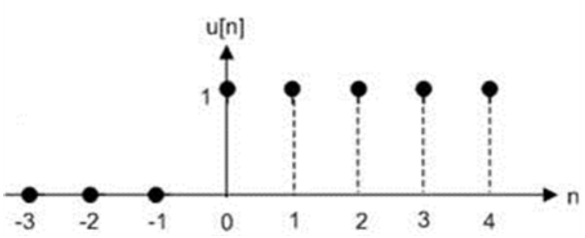
A continuous-time unit step function $u(t)$ is shown below:

$$u(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases} \qquad (1.2)$$

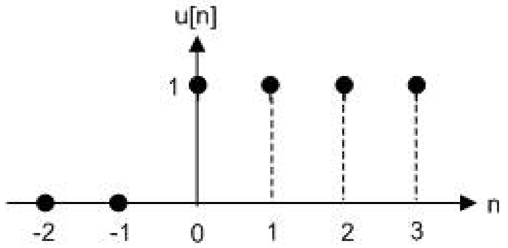


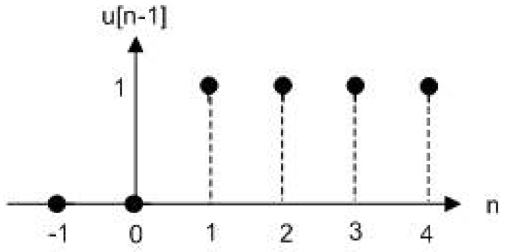
Note that the unit step is discontinuous at t = 0. Its samples $u[n] = u(t)|_{t=nT}$ form the discrete-time signal and defined by

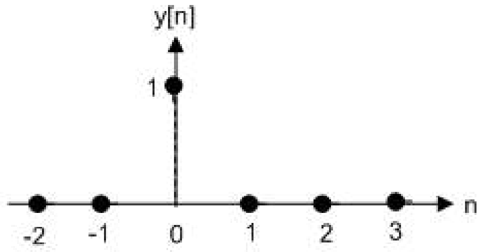
$$u[n]=\begin{cases}1 & n\geq 0\\ 0 & n<0\end{cases} \qquad (1.3)$$



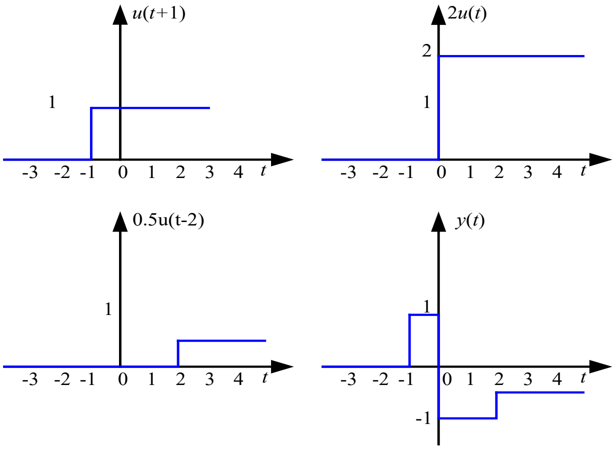
Example: Sketch the wave form: $y[n]=u[n]-u[n-1]$







Example: Sketch the waveform for $y(t)=u(t+1)-2u(t)+0.5u(t-2)$



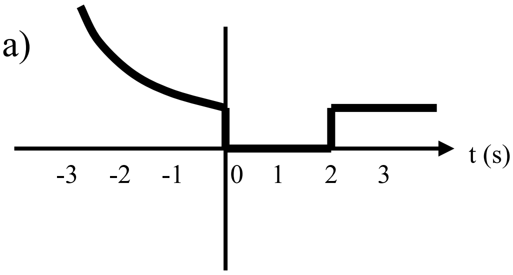
Exercise:

1) Sketch $x(t) = e^{-3t}[u(t) - u(t-2)]$

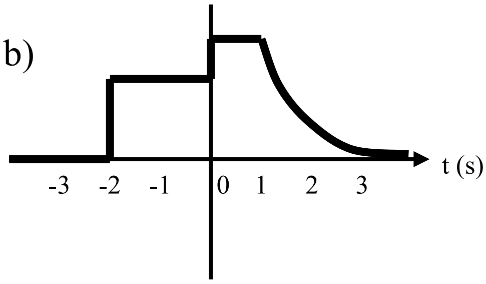
2) Which of the following depicts

$$y(t) = u(t+2) - 2u(t) + 2e^{-(t-1)}u(t-1)$$

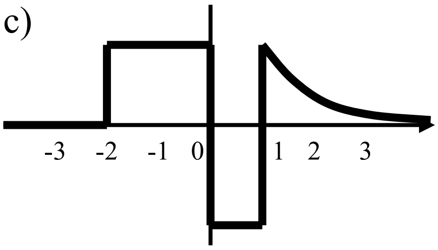
a)



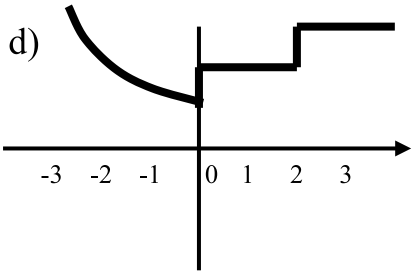
b)



c)



d)



Example:

Continuous-time Signal | Discrete-time signal

1. $x(t) = e^{at}$ $\xrightarrow[t=nT]{}$ $x[n] = e^{anT}$

time (t)

Sample number (n) [0,1,2,3,…]

Sampling Period (T)

$f_s = \frac{1}{T} Hz$ ← sampling frequency

2. $x(t) = 10e^{-t} - 5e^{-0.5t}$ $\xrightarrow[t=nT]{}$ $x[n] = 10^{-nT} - 5e^{-0.5nT}$

sample number (n)

3. $x(t) = A\cos(\omega_a t)$ $\xrightarrow[t=nT]{}$ $x[n] = A\cos(\omega_a nT)$

Analogue Frequency (ω_a) in radians

$\omega_a = 2\pi f_a$

$$= A\cos(2\pi f_a n \frac{1}{f_s})$$

$$= A\cos(2\pi \frac{f_a}{f_s} n)$$

$$x[n] = A\cos(n\theta)$$

$$where \quad \theta = 2\pi \frac{f_a}{f_s} = \omega_a T$$

$$\theta : digital\ frequency(rad)$$

$$-\pi \leq \theta \leq \pi$$

Exercise: An analog signal $x(t) = 14\sin(5000\pi t)$ is sampled at a rate of 6 kHz. The resulting digital signal, $x[n]$, is given by:

a) $\sin[2\pi \frac{5}{6} n]$

b) $14\sin[\frac{5\pi}{6} n]$

c) $14\sin[1000\pi n]$

d) $14\sin[11000\pi n]$

Periodic Signals

An important class of signals is the periodic signals. A periodic continuous-time signal $x(t)$ has the property that there is a positive value of P for which

$$x(t) = x(t+P) \qquad (1.4)$$

for all values of t. In other words, a periodic signal has the property that is unchanged by a time shift of P. In this case we say $x(t)$ is periodic with period P.

Example

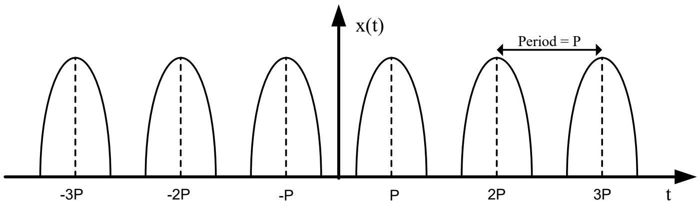
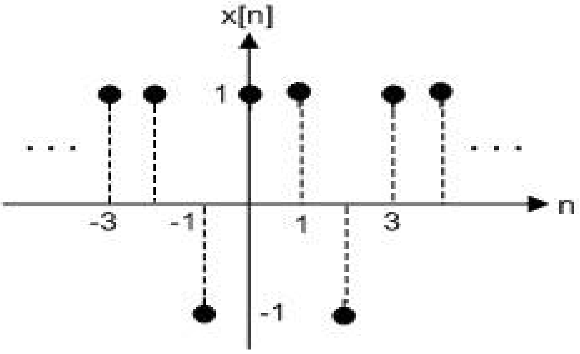


Figure 1.6: An example of a continuous-time periodic signal

Periodic signals are defined analogously in discrete time. A discrete-time signal $x[n]$ is periodic with period N, where N is a positive integer, if for all values of n.

$$x[n] = x[n+N]$$

Example:



x[n] with Period, N = 3 samples

1.2.2 Discrete-Time Sinusoidal Signals

A continuous-time sinusoidal signal is given by

$$x(t) = A\sin(\omega_a t) = A\sin(2\pi f_a t) \qquad (1.6)$$

f_a : analogue frequency

A discrete - time sinusoidal signal may be expressed as

$$x[n] = x(t)|_{t=nT} = x(nT)$$

$$x[n] = A\sin(n\omega_a T) \quad = A\sin(2\pi \frac{f_a}{f_s} n)$$

$$x[n] \quad = A\sin(n\theta) \qquad (1.7)$$

Sampling frequency: $f_s = \frac{1}{T}$

θ - Digital frequency: $\theta = 2\pi \frac{f_a}{f_s} = \omega_a T$ (1.8)

A discrete-time signal is said to be periodic with a period length *N*, if *N* is the smallest integer for which

$$x[n+N] = x[n]$$

$$\therefore A\sin((n+N)\theta) = A\sin(n\theta)$$

which can only be satisfied for all *n* if

$N\theta = 2\pi k$ (where *k* is an arbitrary integer)

$$\therefore N = \frac{2\pi k}{\theta} = \frac{2\pi k}{2\pi \frac{f_a}{f_s}} \longleftarrow \text{see eq. (1.8)}$$

$$N = \frac{f_s}{f_a} k \qquad (1.9)$$

For example, f_a = 1000Hz and f_s = 8000 Hz then $N=f_a / f_s=8$ samples (assume k is the smallest positive integer, i.e. k=1)
The following diagram depicts the cosine wave $x[n] = \cos\left(\frac{2\pi n}{12}\right)$ with period N = 12 samples.

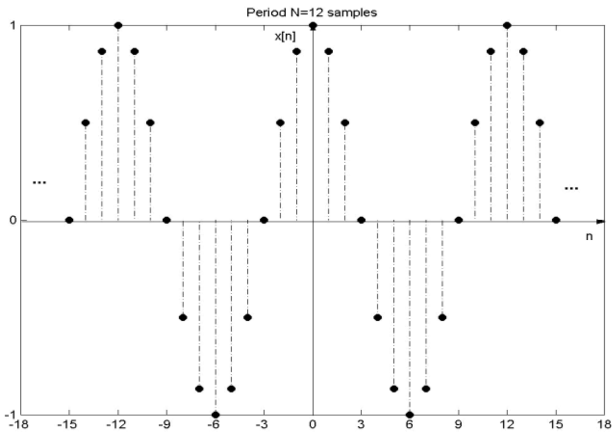


Figure 1.7 An example of a periodic sinusoidal sequence

Example:

Determine the fundamental period of $x\,[n] = 10\cos\left(\frac{2\pi}{15}n + \frac{\pi}{3}\right)$

Digital frequency: $\theta = \frac{2\pi}{15}$

Recall (equation 1.9): N $= \frac{2\pi k}{\theta}$; k=1 is the smallest positive integer

$\therefore$ $N = \frac{2\pi\times1}{2\pi/15}$ = 15 samples

Example:

The sinusoidal signal x[n] has fundamental period N=10 samples. Determine the smallest θ for which x[n] is periodic:

$$\theta = \frac{2\pi k}{N} = \frac{2\pi}{10}k$$

Smallest value of θ is obtained when k = 1

$$\therefore \theta = \frac{2\pi}{10} = \frac{\pi}{5} radians / cycle$$

1.2.3 Discrete-time Exponential Signals

In discrete time, it is common practice to write a real exponential signal as

$$x[n] = c\alpha^n \qquad (1.10)$$

If c and α are real and if $|\alpha|>1$ the magnitude of the signal grows exponentially with n, while if $|\alpha|<1$ we have decaying exponential.

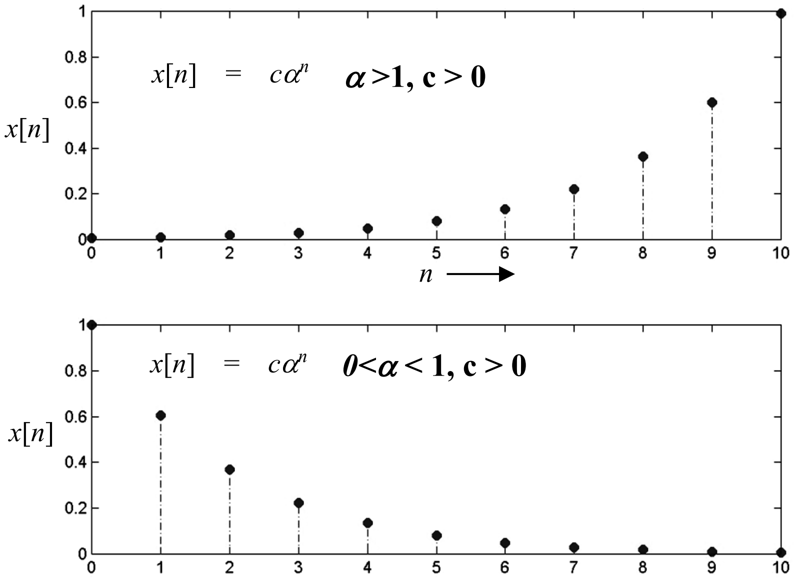
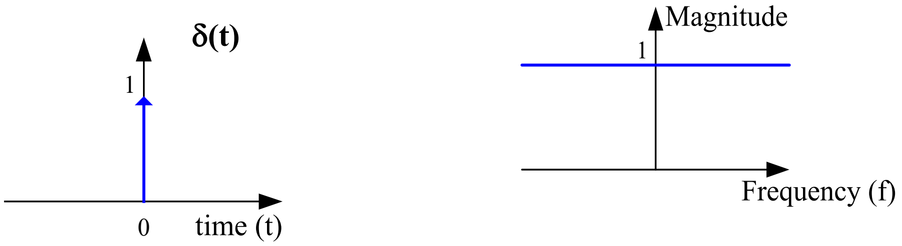


Figure 1.8: Examples of discrete-time exponential signals.

1.2.4 The Unit Impulse

An important concept in the theory of linear systems is the continuous time unit impulse function. This function, known also as the Dirac delta function is denoted by $\delta(t)$ and is represented graphically by a vertical arrow.

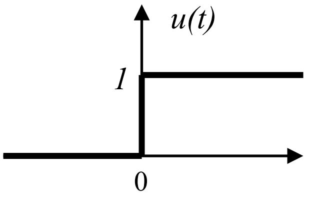


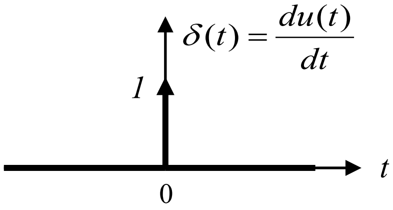
(a) (b)

Figure 1.9 Characteristics of impulse response function (a) in time domain and (b) in the frequency domain

The impulse function $\delta(t)$ is the derivative of the step function $u(t)$.

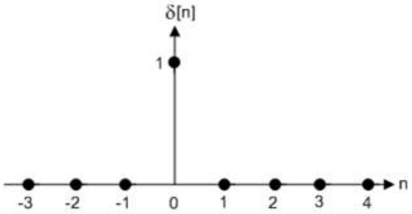
$$\boxed{\delta(t) = \frac{du(t)}{dt} \qquad (1.11)}$$





The discrete-time unit impulse function $\delta[n]$ is defined in a manner similar to its continuous time counterpart. We also refer $\delta[n]$ as the unit sample.

$$\boxed{\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \qquad (1.12)}$$



Exercise:

1) Which of the following signals is not periodic?

   a) x[n] = sin[πn/9]
   b) x[n] = sin[nπ^2]
   c) x[n] = cos[πn^2/15]
   d) x[n] = sin[πn/5 + π] + cos[πn/10 – π]

2) A discrete-time signal is given by x[n] = 3cos[**Error! Bookmark not defined.**$\frac{4\pi}{34}$n + **Error! Bookmark not defined.**$\frac{\pi}{12}$]. The digital frequency (θ) and fundamental period (N) are given by:

   a) $\frac{4\pi}{34}$ and 4 samples
   b) **Error! Bookmark not defined.**$\frac{\pi}{12}$ and 34 samples
   c) $\frac{4\pi}{34}$n and 12 samples

d) $\frac{2\pi}{17}$ and 17 samples

1.2.5 Simple Manipulations of Discrete-Time Signals

A signal $x[n]$ may be shifted in time by replacing the independent variable n by n-k where k is an integer.

If $k>0 \Rightarrow$ the time shift results in a **delay** of the signal by k samples [ie. shifting a signal to the right]

If $k<0 \Rightarrow$ the time shift results in an **advance** of the signal by k samples. [ie. shifting a signal to the left]

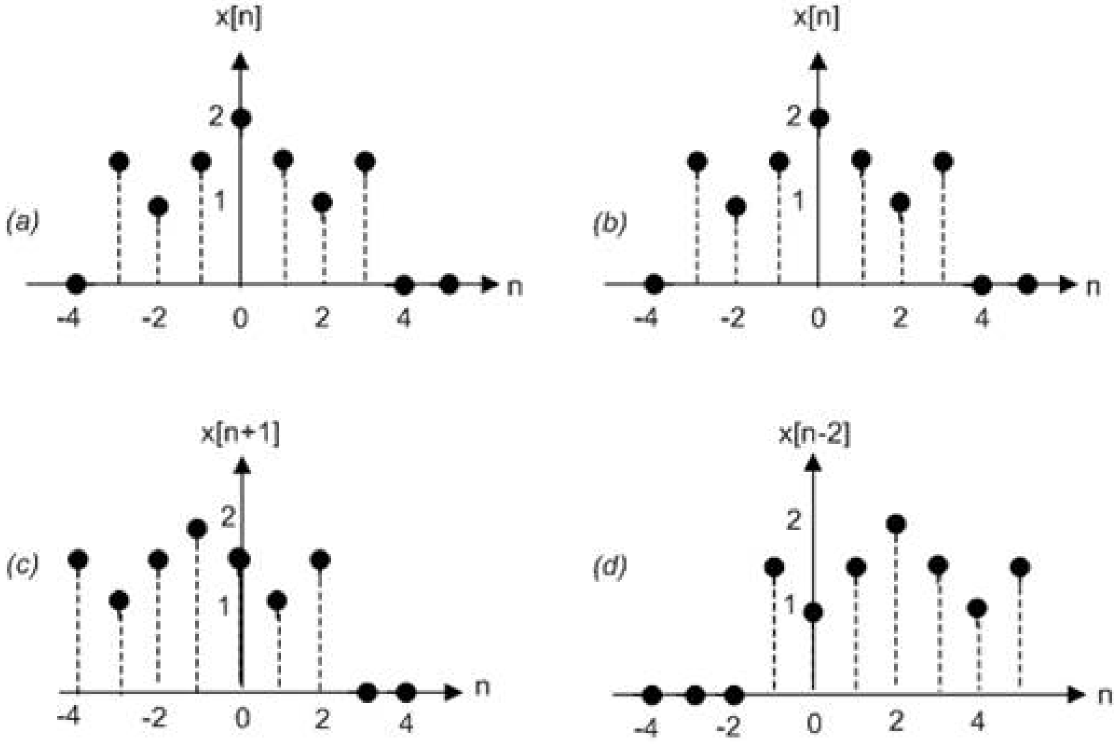


Figure 1.10: (a,b) Original signal $x[n]$*, (c)* $x[n]$ *is advanced by 1 sample, (d)* $x[n]$ *is delayed by 2 samples*

1.3 Systems

A continuous-time system is one whose input $x(t)$ and output $y(t)$ are continuous time functions related by a rule as shown in Figure 1.11.

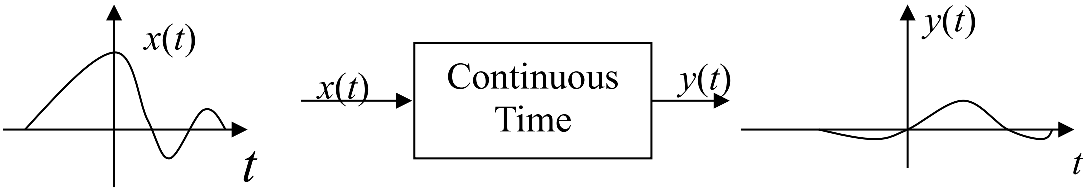


Figure 1.11: General representation of continuous-time systems.

A discrete system is one whose input $x[n]$ and output $y[n]$ are discrete time function related by a rule as shown in Figure 1.12.

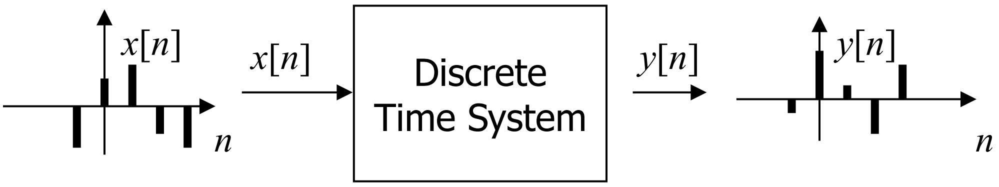
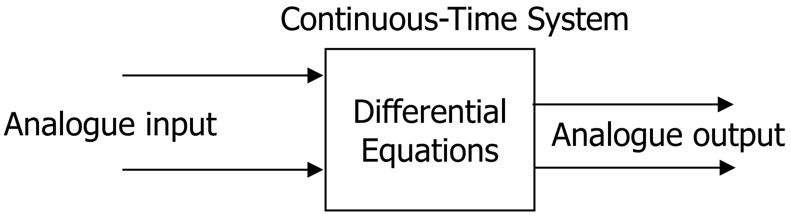
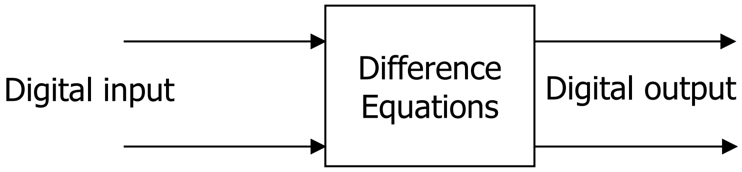


Figure 1.12: General representation of discrete-time systems.

An important mathematical distinction between continuous-time and discrete-time systems is the fact that the former are characterized by differen**tial** equations whereas the latter are characterized by differen**ce** equations.


Continuous-Time System



Discrete-Time System

Example: The RC circuit (known as a low-pass filter or an integrator) shown in Figure 1.13 is a continuous-time system.

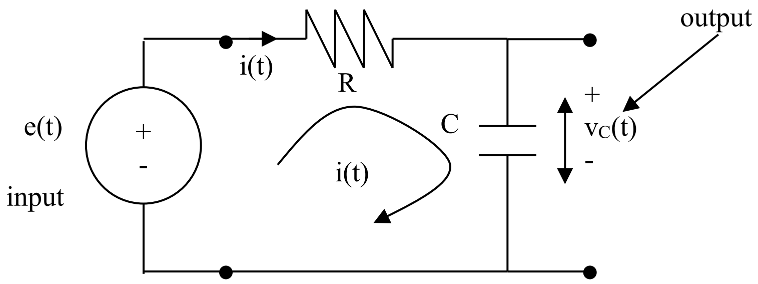


Figure 1.13: A diagram of RC circuit as an example of continuous-time systems.

If we regard $e(t)$ as the <u>input signal</u> and $v_C(t)$ as the <u>output signal</u>, we obtain using simple circuit analysis

$$\frac{dv_C(t)}{dt} + \frac{1}{RC} v_C(t) = \frac{1}{RC} e(t) \qquad (1.13)$$

From equation (1.13), a discrete -time system can be developed as follows: If the sampling period T is sufficiently small, (see figure 1.14)

$$\left.\frac{dv_C(t)}{dt}\right|_{t=nT} = \frac{v_C(nT) - v_C(nT-T)}{T} \qquad (1.14)$$

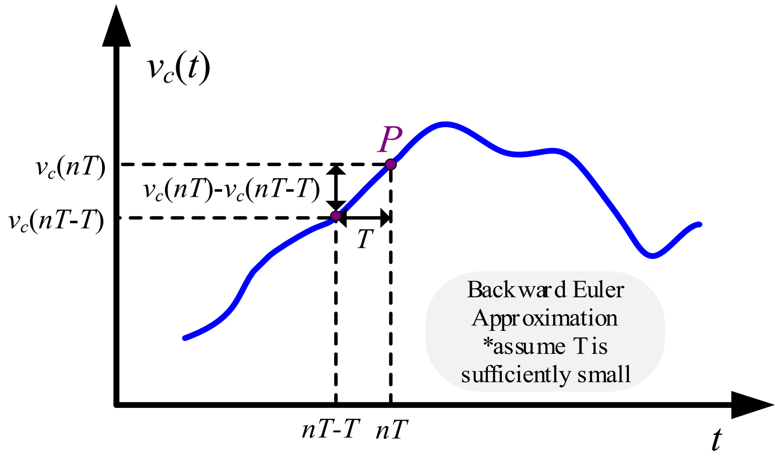


Figure 1.14: An approximation of discrete-time systems from the continuous-time systems.

By substituting equation (1.14) into (1.13) and replacing *t* by *nT*, we obtain:

$$\frac{v_C(nT)-v_C(nT-T)}{T}+\frac{1}{RC}v_C(nT)=\frac{1}{RC}e(nT)$$

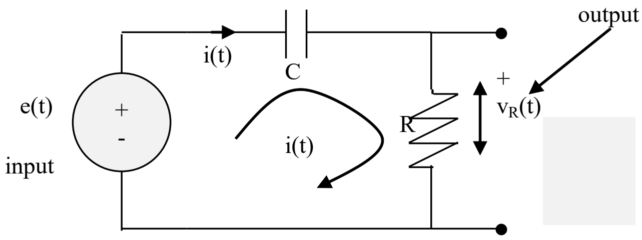
The difference equation is:

$$\frac{v_C[n]-v_C[n-1]}{T}+\frac{1}{RC}v_C[n]=\frac{1}{RC}e[n]$$

$$\boxed{\underset{\overset{\uparrow}{\text{output}}}{v_C[n]}=\frac{RC}{RC+T}\underset{\overset{\uparrow}{\text{previous output}}}{v_C[n-1]}+\frac{T}{RC+T}\underset{\overset{\uparrow}{\text{input}}}{e[n]}\quad(1.10)}\ \text{difference equation}$$

Exercise:

The RC circuit, known as a high-pass filter or a differentiator, is shown below. Derive a difference equation relating the output and input of the circuit.



CHAPTER 1: PROBLEM SHEET 1

Q1. Sketch the following:

- a) $x(t) = u(t-3) - u(t-5)$
- b) $y[n] = u[n+3] - u[n-10]$
- c) $x(t) = e^{2t}u(-t)$
- d) $y[n] = u[-n]$
- e) $h[n] = 2\delta[n+1] + 2\delta[n-1]$
- f) $h[n] = u[n]$, $p[n] = h[-n]$; $q[n] = h[-1-n]$, $r[n] = h[1-n]$

Q2.

a) Consider a discrete-time sequence $x[n] = \cos\left[n\frac{\pi}{8} + \frac{\pi}{5}\right]$

Determine the fundamental period of $x[n]$. **Ans: 16 samples**

b)

i) Consider the sinusoidal signal

$x(t) = 10\sin(2\pi f_a t)$ *where* f_a -analogue frequency and t- time,

Write an equation for the discrete time signal $x[n]$. **Ans: x[n]=10 sin(nθ)**

ii) If $f_a = 200$ Hz and sampling frequency, $f_s = 8000$ Hz, determine the fundamental period of $x[n]$. **Ans: N=40**

End of Chapter 1